

KAUFFMAN STATES, BORDERED ALGEBRAS, AND A BIGRADED KNOT INVARIANT

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1. INTRODUCTION

The Alexander polynomial can be given a state sum formulation [6], as a count of certain Kauffman states, each of which contributes a monomial in a formal variable t . In [19], this description is lifted to knot Floer homology [15, 21]: knot Floer homology can be viewed as a chain complex whose generators correspond to Kauffman states. The Kauffman state model for knot Floer homology was rather elusive: the differentials appearing in that description have a holomorphic definition but at the time were not explicitly computed. A much larger model for knot Floer homology was described in [13], where the generators correspond to certain states in a grid diagram. The grid diagram can be used to compute invariants for small knots [1, 4], but computations are limited by the size of the chain complex (which has $n!$ many generators for a grid diagram of size n).

The aim of this article is to construct and study an invariant of knots, $H^-(K)$, with the following properties.

- (H-1) Letting $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $H^-(K)$ is a bigraded module over the polynomial algebra $\mathbb{F}[U]$. That is, there is a vector space splitting $H^-(K) \cong \bigoplus_{d,s} H_d^-(K, s)$, and an endomorphism U of $H^-(K)$ with

$$U: H_d^-(K, s) \rightarrow H_{d-2}^-(K, s-1).$$

- (H-2) If \mathcal{D} is a diagram for K , with a marked edge, then $H^-(K)$ is obtained as the homology of a chain complex $C^-(\mathcal{D})$ associated to the diagram.
- (H-3) The complex $C^-(\mathcal{D})$ is a bigraded chain complex over $\mathbb{F}[U]$, which is freely generated by the Kauffman states of \mathcal{D} .
- (H-4) The graded Euler characteristic of $H^-(K)$ is related to the symmetrized Alexander polynomial $\Delta_K(t)$ of the knot K , as follows: there is an identification of Laurent series in $\mathbb{Z}[t, t^{-1}]$

$$(1.1) \quad \sum_{d,s} (-1)^d \dim H_d^-(K, s) t^s = \frac{\Delta_K(t)}{(1-t^{-1})}.$$

From its description, this invariant comes equipped with a great deal of algebraic structure, similar to the Khovanov and Khovanov-Rozansky categorifications of the Jones polynomial and its generalizations. The structure also makes computations of the invariant for large examples feasible. In this paper, we also give an algebraic proof of invariance, hence giving a self-contained treatment of this invariant. In a

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follow-up paper, we will show that the invariant constructed here is isomorphic to knot Floer homology.

In a little more detail, a *decorated knot diagram* \mathcal{D} for K is an oriented, generic knot projection of K onto \mathbb{R}^2 , together with a choice of a distinguished edge, which meets the infinite region. The projection gives a planar graph G whose vertices correspond to the double-points of the projection of K . Since G is four-valent, there are four distinct quadrants (bounded by edges) emanating from each vertex, each of which is a corner of the closure of some region of $\mathbb{R}^2 \setminus G$. Let m denote the number of vertices of G . Clearly, G divides \mathbb{R}^2 into $m + 2$ regions, one of which is the unbounded one.

Definition 1.1. A Kauffman state (cf. [6]) for a decorated knot projection of K is a map \mathbf{K} that associates to each vertex of G one of the four in-coming quadrants, subject to the following constraints:

- The quadrants assigned by \mathbf{K} to distinct vertices are subsets of distinct bounded regions in $\mathbb{R}^2 \setminus G$.
- The quadrants of the bounded region that meets the distinguished edge are not assigned by \mathbf{K} to any of the vertices in G .

See Figure 1 for a picture.

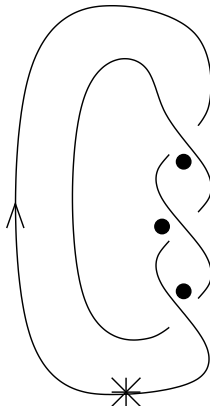


FIGURE 1. **Decorated knot projection for the left-handed trefoil.** The distinguished edge is marked with a star. We have illustrated one of the three Kauffman states for this projection.

It is easy to see that a Kauffman state sets up a one-to-one correspondence between vertices of G and the m connected components of $\mathbb{R}^2 \setminus G$ that do not meet the distinguished edge.

In [6], Kauffman describes the Alexander polynomial of a knot as a sum of monomials in t associated to every Kauffman state. We recall this description (with slight modifications to suit our purposes).

Definition 1.2. Label the four quadrants around each crossing with 0, and $\pm\frac{1}{2}$, according to the orientations as specified in the second line of Figure 2. The Alexander function of a Kauffman state \mathbf{K} , $A(\mathbf{K})$, is the sum, over each crossing, of the

contribution of the quadrant occupied by the state. The Maslov function of a Kauffman state \mathbf{K} is obtained similarly, only now the local contributions are as specified in the third line of Figure 2.

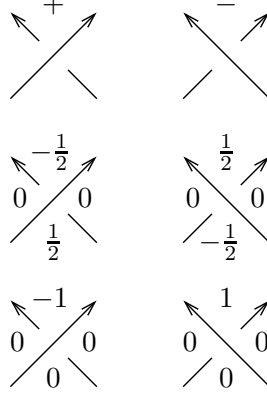


FIGURE 2. **Sign conventions for crossings, and local Alexander and Maslov contributions.** The first row illustrates the chirality of a crossing; the second the Alexander contribution of each quadrant; the third the Maslov contribution of each quadrant.

Let $\mathfrak{S} = \mathfrak{S}(\mathcal{D})$ denote the set of Kauffman states. Kauffman shows that the Alexander polynomial is computed by

$$\Delta_K(t) = \sum_{\mathbf{K} \in \mathfrak{S}} (-1)^{M(\mathbf{K})} t^{A(\mathbf{K})}.$$

(Strictly speaking, Kauffman's description is slightly different; he does not define the integer $M(\mathbf{K})$, only its parity, which is needed for the Alexander polynomial.)

In [19], we gave a description of knot Floer homology as the homology groups of a chain complex whose generators are Kauffman states, with bigradings given by the M and A functions defined above, and whose differential counts pseudo-holomorphic disks. In certain special cases (for example, for alternating knot diagrams), these differentials could be computed explicitly; but in general, their computation remained elusive. (Compare with [2].)

In the present paper, we define $H^-(K)$, which is the homology of a chain complex $C^-(\mathcal{D})$ associated to a diagram. Its generators are Kauffman states, and its differential is described algebraically. The construction will involve decompositions of the knot diagram \mathcal{D} .

Given a generic, oriented knot projection in the plane, equipped with its usual (x, y) coordinates, we will consider the intersection of \mathcal{D} with half-planes $y \geq t$ and $y \leq t$, for generic values of t . Intersections with $y \geq t$ are called *upper partial knot diagrams*, and intersections with $y \leq t$ are called *lower partial knot diagrams*. See Figure 3.

The decorated edge of \mathcal{D} will be one of the points with minimal y value.

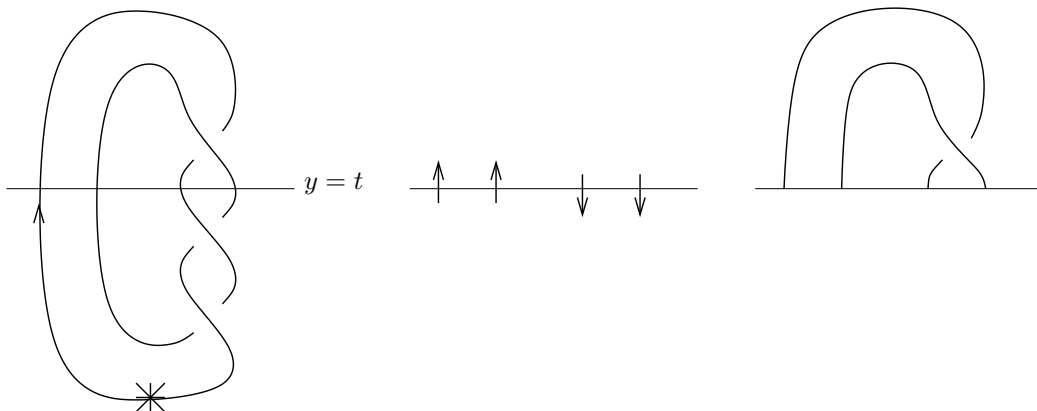


FIGURE 3. **Slicing the trefoil diagram.** Taking the $y = t$ slice for the diagram on the left, we get the line with $2n = 4$ marked points (and orientations) in the middle. At right is the corresponding $y \geq t$ upper knot diagram.

Our knot invariant will be constructed out of invariants for upper and lower knot diagrams. In the spirit of bordered Floer homology [8], we will associate the following algebraic objects to this picture:

- an algebra associated to the intersection of the knot diagram \mathcal{D} with generic lines $y = t$.
- a “type D structure” (in the sense of [8], recalled here in Section 2.4) associated to each (generic) upper diagram.
- a right \mathcal{A}_∞ -module associated to each (generic) lower diagram.

Given this data, the complex $C^-(\mathcal{D})$ is obtained, for generic t , as the tensor product (in the sense of [8], recalled here in Section 2.5) of the invariant associated to the $y \geq t$ upper diagram with the $y \leq t$ lower diagram, where the pairing is taken over the algebra associated to $y = t$ slice.

In more detail, the intersection of the knot diagram \mathcal{D} with a generic horizontal line $y = t$ can be encoded as a line equipped with $2n$ points p_1, \dots, p_{2n} (i.e. where the knot meets the slice), half of which are oriented upwards, and half of which are oriented downwards. Let \mathcal{S} denote the subset of those points oriented upwards. We will define an algebra associated to this configuration. The points subdivide the line into $2n - 1$ bounded intervals, and two unbounded ones. There are certain distinguished *basic idempotents* in the algebra which correspond to choices of n of those bounded intervals. These algebras, and related ones, are constructed in Section 3.

We will describe certain natural modules over this algebra in terms of the following:

Definition 1.3. *An upper Kauffman state for an upper diagram $y \geq t$ is a pair (\mathbf{K}, I) where*

- \mathbf{K} associates to each crossing in the upper diagram one of the four adjacent quadrants.
- I is a basic idempotent for the $y = t$ algebra.

Moreover, these data are required to satisfy the compatibility conditions:

- The quadrants assigned by \mathbf{K} to distinct vertices are subsets of distinct bounded regions in the upper diagram $y \geq t$.
- The unbounded region meets none of the intervals in the idempotent I .
- Each bounded region in the upper diagram $y \geq t$ either:
 - contains a quadrant associated to some vertex by \mathbf{K} , and meets none of the intervals in the idempotent I ; or
 - meets exactly one of the intervals in the idempotent I , and is not associated to any vertex by \mathbf{K} .

It is easy to see that any Kauffman state can be restricted to give an upper Kauffman state on any of its upper diagrams. On the other hand upper Kauffman states on an upper diagram might not extend to give a global Kauffman state.

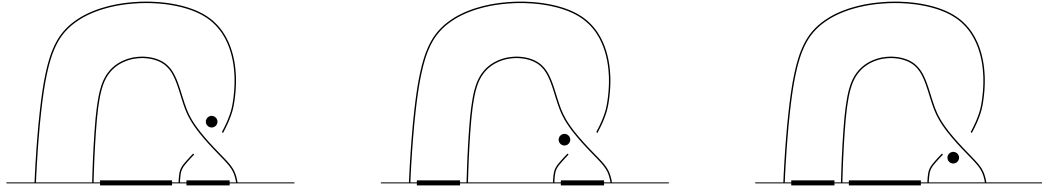


FIGURE 4. **Upper Kauffman states.** We have illustrated here all three upper Kauffman states for the upper diagram obtained from the trefoil diagram in Figure 3. The idempotent I is represented by the pairs of darkened intervals.

The type D structure of an upper diagram is generated by its upper Kauffman states.

We will also describe bimodules for enlarging the upper knot diagram. Specifically, we will associate bimodules to crossings, caps, and cups, in Sections 5, 8, and 9 respectively. These bimodules are generated by certain localized Kauffman states, and the type D structure of an upper diagram is built as a tensor product of the bimodules associated to these various pieces. Tensoring together all the pieces gives the desired chain complex $C^-(\mathcal{D})$ whose generators are Kauffman states.

The topological invariance of $H^-(K)$ is established in Theorem 11.6. The proof proceeds by showing that $H^-(\mathcal{D})$ is invariant under planar isotopies of the knot diagram, and then locally verifying invariance under the three Reidemeister moves. In fact, invariance under Reidemeister moves 2 and 3 are part of the “braid relations” that the crossing bimodules satisfy (see Section 6), while Reidemeister 1 invariance is an easy computation.

The complex $C^-(\mathcal{D})$ is a chain complex over the polynomial ring in one generator U . We can set $U = 0$ to get another chain complex $\widehat{C}(\mathcal{D})$ whose homology $\widehat{H}(K)$ is also a knot invariant. Thus, $\widehat{H}(K)$ corresponds to $\widehat{HFK}(K)$ and $H^-(K)$ corresponds to $\text{HFK}^-(K)$.

Knot Floer homology [15] is an invariant with the above properties, according to [14]. The differentials appearing in the original definition of knot Floer homology

involve analytical choices; algebraic constructions over a larger base ring were given in [16] and [2], and a chain complex with many more generators was given in [13].

The methods of this paper are conceptually similar to the computation of Heegaard Floer homology groups of three-manifolds by factoring mapping classes [9]; but the present constructions are ultimately algebraic in nature, as is the invariance proof we give here; compare [24]. In [17], we will in fact verify that $H^-(K)$ coincides with knot Floer homology $\text{HFK}^-(K)$.

1.1. Organization. In Section 2 we discuss the algebraic preliminaries. Much of this discussion is an adaptation of the algebraic portion of [11]. We will make heavy use throughout of the notions of bimodules of various types (DD , DA , and AA), in the language of [11]. Section 3 we describe the algebras associated to knot diagram slices, along with a canonical (invertible) dualizing bimodule of type DD . In Section 4, we associate natural type DD bimodules to crossings, which are simple to describe. In Section 5 we construct the corresponding type DA bimodules, which are most useful to work with. These bimodules induce a braid group action on the category of modules over our algebras, as verified in Section 6. In Section 7, we construct the type DD bimodules associated to a critical point in the knot diagram, and verify the “trident relation”, which describes how these bimodules interact with nearby crossing bimodules. The critical point bimodules have two kinds of corresponding type DA bimodules: the bimodule associated to a maximum, constructed in Section 8, and the bimodule associated to a minimum, constructed in Section 9. The theory is equipped with several symmetries, collected in Section 10. In Section 11, we construct the knot invariant from the constituent bimodules, and verify its invariance properties. In Section 12 we verify a few basic properties of this invariant.

A bordered theory for tangles in the grid context was developed by Ina Petkova and Vera Vértési [20]; compare also [10]. In a quite different direction, Rumen Zarev [23] constructed a “bordered sutured” invariant that can also be used to study knot Floer homology. Explicit computations of $\widehat{H}(K)$, for three-stranded pretzel knots, were done by Andrew Manion. A version of this construction for singular knots is also studied by Manion [12].

In [18], we give a slight variation on the present construction, together with a sign refinement. The techniques from that paper lead to efficient computations of $H^-(K)$ for large knots. In fact, note that, similar to fast computations for Khovanov homology [3] for non-alternating projections, there are various cancelling differentials in the invariants associated to partial knot diagrams that allow for fast computer calculation. Details will be explained in [18]. As an illustration using the 91-crossing presentation of the K_3 knot from [5], we find that the Poincaré polynomial of $\widehat{H}(K)$, defined by

$$P_K(m, t) = \sum_{d, s} \dim \widehat{H}_d(K, s) m^d t^s,$$

is given by

$$\begin{aligned}
& 2t^4m^6 + t^3(7m^5 + 3m^3 + m) \\
& + t^2(10m^4 + m^3 + 16m^2 + m + 3) \\
& + t(7m^3 + 2m^2 + 39m + 2 + 3m^{-1}) \\
& + (4m^2 + 2m + 53 + 2m^{-1} + 2m^{-2}) \\
& + t^{-1}(7m + 2 + 39m^{-1} + 2m^{-2} + 3m^{-3}) \\
& + t^{-2}(10 + m^{-1} + 16m^{-2} + m^{-3} + 3m^{-4}) \\
& + t^{-3}(7m^{-1} + 3m^{-3} + m^{-5}) + 2t^{-4}m^{-2}
\end{aligned}$$

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2. ALGEBRAIC PRELIMINARIES

We recall some algebraic preliminaries from bordered Floer homology. Further background on \mathcal{A}_∞ algebras can be found in [7]. Most of this material (except Section 2.9) can be found, with more detail, in [11].

2.1. Algebras. In this paper, we will be concerned with differential graded algebras \mathcal{A} (DG algebras) in characteristic 2.

The DG algebra \mathcal{A} is an abelian group equipped with a differential

$$\mu_1 : \mathcal{A} \rightarrow \mathcal{A},$$

and a multiplication map

$$\mu_2 : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A},$$

satisfying the usual compatibility conditions

$$\begin{aligned} \mu_1^2 &= 0; \\ \mu_2 \circ (\mu_1 \otimes \text{Id} + \text{Id} \otimes \mu_1) &= \mu_1 \circ \mu_2; \\ \mu_2 \circ (\mu_2 \otimes \text{Id}) &= \mu_2 \circ (\text{Id} \otimes \mu_2). \end{aligned}$$

(The latter two are the Leibnitz rule and associativity rule respectively.) It is customary to abbreviate $\mu_1(a)$ by da and $\mu_2(a \otimes b)$ by $a \cdot b$.)

Our algebras are strictly unital; i.e. they are equipped with a distinguished element 1 which is a cycle. We will typically think of our algebras as defined over a ground ring \mathbf{k} , which in turn is a direct sum of finitely many copies of $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, equipped with a vanishing differential. This means that there is a distinguished subalgebra of \mathcal{A} , identified with \mathbf{k} , whose unit 1 is also the unit in \mathcal{A} . This means that we can think of \mathcal{A} as a bimodule over \mathbf{k} , μ_1 as a bimodule homomorphism, and $\mu_2 : \mathcal{A} \otimes_{\mathbf{k}} \mathcal{A} \rightarrow \mathcal{A}$ as a bimodule homomorphism.

Note that our algebras will be typically not finitely generated over the ground ring \mathbf{k} .

2.2. Gradings. Our algebras will be equipped with gradings, a *Maslov grading*, which takes values in \mathbb{Z} , and an *Alexander multi-grading* which takes values in some Abelian group $\Lambda = \Lambda_{\mathcal{A}}$, which are compatible with the algebra actions as described presently.

Specifically, \mathcal{A} is equipped with a direct sum splitting

$$\mathcal{A} = \bigoplus_{(d,\ell) \in \mathbb{Z} \oplus \Lambda} \mathcal{A}_{d;\ell}.$$

A non-zero element $a \in \mathcal{A}_{d;\ell}$ is called *homogeneous with grading* (d,ℓ) ; or simply *homogeneous with respect to the grading by $\mathbb{Z} \oplus \Lambda$* , when we do not wish to specify its actual grading. Similarly, if $a \in \bigoplus_{d \in \mathbb{Z}} \mathcal{A}_{d;\ell}$ for some fixed $\ell \in \Lambda$, we say that a is Λ -homogeneous (with grading ℓ).

The compatibility with the algebra actions is given by

$$\begin{aligned} \mu_1 : \mathcal{A}_{d;\ell} &\rightarrow \mathcal{A}_{d-1;\ell} \\ \mu_2 : \mathcal{A}_{d_1;\ell_1} \otimes \mathcal{A}_{d_2;\ell_2} &\rightarrow \mathcal{A}_{d_1+d_2;\ell_1+\ell_2}. \end{aligned}$$

In our present applications, the group $\Lambda_{\mathcal{A}}$ is $(\frac{1}{2}\mathbb{Z})^m \subset \mathbb{Q}^m$.

The algebras considered here will satisfy the following further condition:

Definition 2.1. *We say that the Alexander multi-grading on \mathcal{A} is positive over \mathbf{k} if the following two conditions hold:*

- \mathbf{k} is the set of algebra elements in \mathcal{A} with Alexander multi-grading 0.
- For a_1, \dots, a_m with Alexander multi-gradings $\lambda_1, \dots, \lambda_m$, if $\sum_{i=1}^m \lambda_i = 0$, then each $\lambda_i = 0$.

2.3. Modules. We will consider several kinds of modules over our algebras.

A *right differential module* over \mathcal{A} is a right \mathbf{k} -module M , equipped with maps

$$m_1: M \rightarrow M.$$

and

$$m_2: M \otimes_{\mathbf{k}} \mathcal{A} \rightarrow M.$$

satisfying

$$\begin{aligned} m_1^2 &= 0; \\ m_2 \circ (m_1 \otimes \text{Id} + \text{Id} \otimes \mu_1) &= m_1 \circ m_2; \\ m_2 \circ (m_2 \otimes \text{Id}) &= m_2 \circ (\text{Id} \otimes \mu_2). \end{aligned}$$

We will consider modules that are *strictly unital*, meaning that $m_2(x, 1) = x$ for all $x \in M$. In this case, we can think of m_2 as a right \mathbf{k} -module map $m_2: M \otimes_{\mathbf{k}} \mathcal{A} \rightarrow M$.

Weakening associativity, one naturally arrives at the notion of an \mathcal{A}_{∞} module M . A *right \mathcal{A}_{∞} module* over \mathcal{A} is a right \mathbf{k} -module M , equipped with a sequence of maps

$$m_i: M \otimes_{\mathbf{k}} \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^{i-1} \rightarrow M$$

for $i \geq 1$ satisfying the following strict unitality conditions:

- (1) $m_2(x \otimes 1) = x$ for all $x \in M$
- (2) $m_i(x \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ if $i > 2$ and there is some $1 \leq j \leq i-1$ with $a_j = 1$;

and a compatibility condition which is perhaps best phrased in terms of the bar construction. (See Equation (2.1) below.) Define

$$\mathcal{T}^*(\mathcal{A}) = \bigoplus_{i=0}^{\infty} \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^i,$$

with the convention that the 0^{th} tensor product of \mathcal{A} is \mathbf{k} . This is a chain complex, with a differential induced by μ_1 and μ_2 ; i.e.

$$\begin{aligned} \underline{d}(a_1 \otimes \cdots \otimes a_i) &= \sum_{j=1}^i a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_1(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_i \\ &\quad + \sum_{j=1}^{i-1} a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_2(a_j \otimes a_{j+1}) \otimes \cdots \otimes a_i, \end{aligned}$$

with the understanding that \underline{d} vanishes on $\mathcal{T}^0(\mathcal{A})$. Consider

$$\mathcal{T}^*(M) = M \otimes_{\mathbf{k}} \mathcal{T}^*(\mathcal{A})$$

The maps m_i ($i \geq 1$) induce a map

$$\underline{m}: \mathcal{T}^*(M) \rightarrow \mathcal{T}^*(M);$$

by the formula

$$\underline{m}(x \otimes a_1 \otimes \cdots \otimes a_i) = \sum_{j=0}^i m_{j+1}(x \otimes a_1 \otimes \cdots \otimes a_j) \otimes a_{j+1} \otimes \cdots \otimes a_i$$

The compatibility condition is equivalent to the condition that

$$(2.1) \quad \underline{m}(\underline{m}(x \otimes \underline{a})) + \underline{m}(x \otimes \underline{d}(a)) = 0;$$

or equivalently, the map $\underline{d}_M: \mathcal{T}^*(M) \rightarrow \mathcal{T}^*(M)$ defined by

$$\underline{d}_M(x \otimes \underline{a}) = \underline{m}(x \otimes \underline{a}) + x \otimes \underline{d}(\underline{a})$$

is a differential. A right differential module is a right \mathcal{A}_∞ module with $m_i = 0$ for all $i \geq 3$.

Left differential modules and \mathcal{A}_∞ modules are defined analogously (though in adherence with the conventions laid down in [8], our \mathcal{A}_∞ modules will typically be right modules).

Gradings are as follows. The modules M we consider will typically have a Maslov grading by \mathbb{Z} , and a further Alexander multi-grading set S , which is a set S with an action by $\Lambda_{\mathcal{A}}$. That is, there is a direct sum splitting (as \mathbf{k} -modules):

$$M = \bigoplus_{d \in \mathbb{Z}, s \in S} M_{d;s}$$

and we assume that each summand $M_{d;s}$ is a finitely generated \mathbf{k} -module.

The actions m_i will be graded as follows.

$$m_i: M_{d_0;s} \otimes \mathcal{A}_{d_1;\ell_1} \otimes \cdots \mathcal{A}_{d_{i-1};\ell_{i-1}} \rightarrow M_{i-2+\sum_{j=0}^{i-1} d_j; s+\sum_{j=1}^{i-1} \ell_j}.$$

We will typically record the algebra \mathcal{A} as a subscript for the \mathcal{A}_∞ -module M , writing $M_{\mathcal{A}}$ if M is a right \mathcal{A}_∞ -module, and $_{\mathcal{A}}M$ if M is a left \mathcal{A}_∞ -module.

Given two \mathcal{A}_∞ modules $M_{\mathcal{A}}$ and $N_{\mathcal{A}}$, a *morphism* from $M_{\mathcal{A}}$ to $N_{\mathcal{A}}$ is a sequence

of \mathbf{k} -module maps $\{\phi_i: M_{\mathcal{A}} \otimes_{\mathbf{k}} \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^{i-1} \rightarrow N_{\mathcal{A}}\}_{i \geq 1}$. A morphism naturally induces a map

$$\underline{\phi}: M_{\mathcal{A}} \otimes \mathcal{T}^*(\mathcal{A}) \rightarrow N_{\mathcal{A}} \otimes \mathcal{T}^*(\mathcal{A}).$$

by

$$\underline{\phi}(x \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^n \phi_{i+1}(x \otimes a_1 \otimes \cdots \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

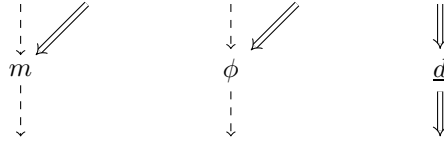
When this induced map is a chain map, we say that the morphism is a *homomorphism*. More generally, the space of morphisms can be given a differential, so that

$$d_{\text{Mor}}(\underline{\phi}) = \underline{d}_N \circ \underline{\phi} + \underline{\phi} \circ \underline{d}_M.$$

Let $\mathfrak{Mod}_{\mathcal{A}}$ resp. ${}_{\mathcal{A}}\mathfrak{Mod}$ denote the category of right resp. left \mathcal{A}_{∞} modules over \mathcal{A} . This is a differential category (i.e. the morphism spaces are chain complexes). Specifically, given $M_{\mathcal{A}}, N_{\mathcal{A}} \in \mathfrak{Mod}_{\mathcal{A}}$, let $\text{Mor}_{\mathcal{A}}(M, N)$ denote the chain complex whose elements are maps $\{\phi_i: M \otimes \mathcal{A}^{i-1} \rightarrow N\}_{i \geq 1}$, with differential given by

$$\begin{aligned} (d\phi)_i(x, a_1, \dots, a_{i-1}) &= \sum_{j=1}^i \phi_{i-j+1}(m_j^M(x, a_1, \dots, a_{j-1}), a_j, \dots, a_{i-1}) \\ &\quad + \sum_{j=1}^i m_{i-j+1}^N(\phi_j(x, a_1, \dots, a_{j-1}), a_j, \dots, a_{i-1}) \\ &\quad + \sum_{j=1}^{i-1} \phi_i(x, a_1, \dots, \mu_1(a_j), \dots, a_{i-1}) \\ &\quad + \sum_{j=1}^{i-2} \phi_{i-1}(x, a_1, \dots, \mu_2(a_j, a_{j+1}), \dots, a_{i-1}). \end{aligned}$$

There is a convenient graphical representation of formulas such as the one above. We represent elements of M by dashed arrows, elements of $\mathcal{T}^*(\mathcal{A})$ by doubled arrows, and various maps between them by labelled nodes. For instance, the map $m: M \otimes \mathcal{T}^*(\mathcal{A}) \rightarrow M$, the morphism $\phi: M \otimes \mathcal{T}^*(\mathcal{A}) \rightarrow N$, and the differential $\underline{d}: \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{A})$ are represented by the pictures



Using the map $\Delta: \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{A}) \otimes \mathcal{T}^*(\mathcal{A})$ defined by

$$\Delta(a_1 \otimes \cdots \otimes a_j) = \sum_{i=0}^j (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_j),$$

we can express the differential on ϕ defined above as:

Let $M_{\mathcal{A}}$ be a right \mathcal{A}_{∞} -module over \mathbf{k} , with a (\mathbb{Z} -valued) Maslov grading and an Alexander multi-grading with values in S . We can form the *opposite module* ${}_{\mathcal{A}}\overline{M}$, which is a space of maps from M to \mathbb{F} , also equipped with a Maslov grading and a grading by S ,

$$(2.2) \quad {}_{\mathcal{A}}\overline{M} = \bigoplus_{d \in \mathbb{Z}, s \in S} \text{Hom}_{\mathbb{F}}(M_{-d; -s}, \mathbb{F}),$$

with action specified as follows. For fixed $\phi \in \text{Hom}_{\mathbf{k}}(M, \mathbb{F})$ and $a_1 \dots, a_{i-1} \in \mathcal{A}$, let $\overline{m}_i(a_1 \otimes \dots \otimes a_{i-1} \otimes \phi)$ be the homomorphism from M to \mathbb{F} whose evaluation on x is given by $\phi(m_i(x \otimes a_{i-1} \otimes \dots \otimes a_1))$. (Note that the tensor factors appearing here are over $\mathbb{Z}/2\mathbb{Z}$; when M is strictly unital, we can think of the tensor factors as taken over \mathbf{k} ; with the understanding that in $a_{i-1} \otimes \dots \otimes a_1$, the bimodule actions of \mathbf{k} on the a_i are also opposites.)

2.4. Type D structures. A *left type D structure* over \mathcal{A} is a left \mathbf{k} -module X , equipped with a \mathbf{k} -linear map

$$\delta^1: X \rightarrow \mathcal{A} \otimes_{\mathbf{k}} X,$$

satisfying the compatibility condition

$$(\mu_2 \otimes \text{Id}_X) \circ (\text{Id}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \text{Id}) \circ \delta^1 = 0.$$

A right type D structure is defined analogously. The maps will be drawn as

and the structure relation is drawn

As in the case of modules, our type D structures will be given a grading by \mathbb{Z} (the Maslov grading) and a further Alexander grading set $S = S_X$, a set with an action by $\Lambda_{\mathcal{A}}$. This means that X is equipped with a direct sum splitting

$$X = \bigoplus_{d \in \mathbb{Z}; s \in S} X_{d;s},$$

where each $X_{d;s}$ is a \mathbf{k} -module. (In fact, in the cases of relevance to us, X will be finitely generated as a \mathbf{k} -module.) The actions respect these gradings, in the sense that

$$\delta^1: X_{d;s} \rightarrow \bigoplus_{d_0+d_1=d-1; \ell_0+s_1=s} \mathcal{A}_{d_0; \ell_0} \otimes X_{d_1; s_1}.$$

We abbreviate this, writing

$$\delta^1: X_{d;s} \rightarrow (\mathcal{A} \otimes X)_{d-1;s}.$$

In practice, the grading set S will be a quotient of $\Lambda_{\mathcal{A}}$.

A left type D structure X induces a left differential module $\mathcal{A} \boxtimes X$ in a natural way. As a left \mathcal{A} -module, the space is $\mathcal{A} \otimes_{\mathbf{k}} X$; i.e. given $a \in \mathcal{A}$ and $b \otimes \mathbf{x} \in \mathcal{A} \otimes_{\mathbf{k}} X$, we define

$$m_2(a, b \otimes x) = \mu_2(a, b) \otimes x.$$

The operator

$$m_1 = \mu_1 \otimes \text{Id}_X + (\mu_2 \otimes \text{Id}_X) \circ (\text{Id}_{\mathcal{A}} \otimes \delta_X^1)$$

which can be graphically represented by

induces differential on $\mathcal{A} \otimes_{\mathbf{k}} X = \mathcal{A} \boxtimes X$; i.e. $m_1^2 = 0$. It is straightforward to check that the differential and the \mathcal{A} -action satisfy a Leibniz rule; i.e. $\mathcal{A} \boxtimes X$ is a differential \mathcal{A} -module. (This is the special case of a more general construction: it is the tensor product of \mathcal{A} , viewed as a bimodule over itself, with the type D structure X .)

Conversely, let M be a left DG-module over \mathcal{A} , that splits as a left \mathcal{A} -module as a direct sum of modules that are isomorphic to left ideals in \mathcal{A} by elements of \mathbf{k} ; i.e.

$$(2.3) \quad M = \bigoplus_{g \in G} \mathcal{A} \cdot \mathbf{Y}(g)$$

for some finite set G , and a map $\mathbf{Y}: G \rightarrow \mathbf{k}$. There is a \mathbf{k} -submodule of M ,

$$X = \bigoplus_{g \in G} \mathbf{k} \cdot \mathbf{Y}(g).$$

Restricting the differential m_1 to X gives a map $\delta^1: X \rightarrow \mathcal{A} \otimes X = M$; the hypothesis that $m_1^2 = 0$ is equivalent to the condition that δ^1 determines a type D structure.

We will typically record the algebra as a superscript for a type D structure, writing ${}^{\mathcal{A}}X$ to denote a left type D structure X over \mathcal{A} . There is an analogous notion of right type D structures; for a right type D structure, we record the algebra (as a superscript) on the right, e.g. writing $X^{\mathcal{A}}$.

Analogously, we let ${}^{\mathcal{A}}\mathfrak{Mod}$ and $\mathfrak{Mod}^{\mathcal{A}}$ denote the category of left resp. right type D structures. For this category, $\text{Mor}({}^{\mathcal{A}}P, {}^{\mathcal{A}}Q)$ is defined to be the chain complex of maps

$$h^1: P \rightarrow \mathcal{A} \otimes_{\mathbf{k}} Q,$$

where the differential is specified by

$$d(h^1) = (\mu_1^{\mathcal{A}} \otimes \text{Id}_Q) \circ h^1 + (\mu_2^{\mathcal{A}} \otimes \text{Id}_Q) \circ (\text{Id}_{\mathcal{A}} \otimes h^1) \circ \delta_P^1 + (\mu_2^{\mathcal{A}} \otimes \text{Id}_Q) \circ (\text{Id}_{\mathcal{A}} \otimes \delta_Q^1) \circ h^1.$$

There is also a composition map, which is a chain map $\text{Mor}({}^{\mathcal{A}}P, {}^{\mathcal{A}}Q) \otimes \text{Mor}({}^{\mathcal{A}}Q, {}^{\mathcal{A}}R) \rightarrow \text{Mor}({}^{\mathcal{A}}P, {}^{\mathcal{A}}R)$, defined by taking $f^1 \otimes g^1$ to

$$(2.4) \quad (f \circ g)^1 = (\mu_2 \otimes \text{Id}_Z) \circ (\text{Id}_{\mathcal{A}} \otimes g^1) \circ f^1.$$

If ${}^{\mathcal{A}}X$ is a left type D structure, we can form the *opposite* type D structure as follows.

As a \mathbf{k} -module, $\overline{X}^{\mathcal{A}}$ is a space of vector space maps from M to \mathbb{F} :

$$\overline{X}^{\mathcal{A}} = \bigoplus_{d \in \mathbb{Z}; s \in S} \text{Hom}_{\mathbb{F}}(X_{-d; -s}, \mathbb{F}).$$

This inherits naturally the structure of a right \mathbf{k} -module: the action of $\iota \in \mathbf{k}$ on $\phi: M \rightarrow \mathbb{F}$ is the map that sends x to $\phi(\iota \cdot x)$.

The requisite map

$$\overline{\delta}^1: \overline{X}^{\mathcal{A}} \rightarrow \overline{X}^{\mathcal{A}} \otimes \mathcal{A}$$

is adjoint to the map δ^1 for ${}^{\mathcal{A}}X$; i.e. given $\phi \in {}^{\mathcal{A}}X$, define $\overline{\delta}^1(\phi) \in \text{Hom}(X, \mathbb{F}) \otimes_{\mathbf{k}} \mathcal{A} \subset \text{Hom}(X, \mathcal{A})$ to be the map that sends x to $\langle \delta^1 x, \phi \rangle$, where

$$\langle \cdot, \cdot \rangle: \mathcal{A} \otimes X \otimes \text{Hom}(X, \mathbb{F}) \rightarrow \mathcal{A}$$

is induced by the evaluation map.

2.5. Tensor products. We recall the pairing between \mathcal{A}_∞ modules and type D structures from [8], which in fact can be thought of as a model for the derived tensor product. (See for example [11, Proposition 2.3.18])

Fix first a type D structure ${}^{\mathcal{A}}X$. There are maps for integers $j \geq 0$ with

$$(2.5) \quad \delta^j : X \mapsto \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^j \otimes_{\mathbf{k}} X,$$

with the following inductive definition:

- δ^0 is the identity map;
- δ^1 is as specified by the type D structure;
- and finally,

$$\delta^j = (\text{Id}_{\mathcal{A}^{\otimes j-1}} \otimes \delta^1) \circ \delta^{j-1}.$$

The sum $\sum_{j=0}^{\infty} \delta^j$ is notated

(Note that in general, the image of δ is contained in $(\prod_{m=0}^{\infty} \mathcal{T}^*(\mathcal{A})) \otimes X$.)

Recall that if M is a right \mathcal{A}_∞ -module and X is a left type D structure, then under suitable circumstances, we can form the tensor product $M \boxtimes X$. This is a chain complex whose underlying vector space is $M \otimes_{\mathbf{k}} X$, and whose differential is given by

$$\partial(p \otimes x) = \sum_{j=0}^{\infty} (m_{j+1} \otimes \text{Id}_X) \circ (p \otimes \delta^j(x))$$

In general, the sum appearing in the definition of ∂ has infinitely many terms. The suitable circumstances needed to define ∂ are those where the above sum is finite. For instance, if the module M has the property that for all $p \in M$, $m_{j+1}(p, a_1, \dots, a_j) = 0$ for all sufficiently large j and any sequence a_1, \dots, a_j , then the sum is guaranteed to be finite. Such an \mathcal{A}_∞ module is called a *bounded \mathcal{A}_∞ module*. Similarly, finiteness is guaranteed if X has the property that for all

$x \in X$, there is a j with the property that $\delta^j x = 0$ for all sufficiently large j . Such a type D structure is called a *bounded type D structure*. To recapitulate, $M \boxtimes X$ exists if either $M_{\mathcal{A}}$ or ${}^{\mathcal{A}}X$ is bounded.

Let M be a right \mathcal{A}_{∞} module with Alexander grading set S and X a left type D structure with Alexander grading set T . Then the tensor product $M \boxtimes X$ is naturally graded by the product of \mathbb{Z} (the Maslov grading) and the Alexander grading set $S \times_{\Lambda} T = (S \times T)/\Lambda_{\mathcal{A}}$.

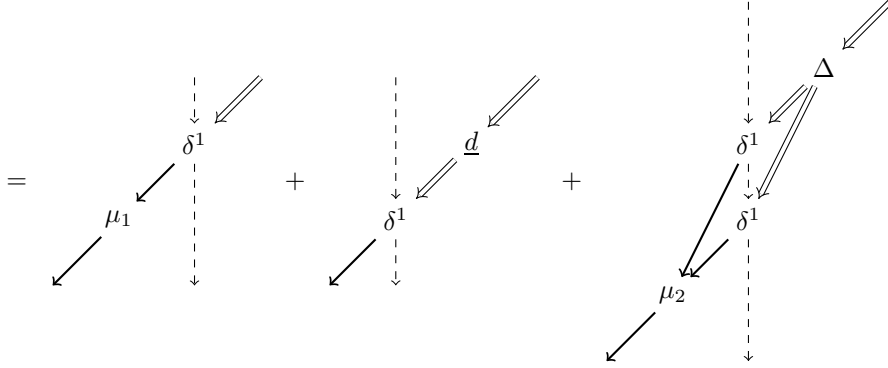
2.6. Bimodules. If \mathcal{A} and \mathcal{B} are two differential graded algebras over ground rings \mathbf{j} and \mathbf{k} respectively, a *left/left type DD bimodule* is a type D structure over the tensor product $\mathcal{A} \otimes \mathcal{B}$. In particular, it is a left module over $\mathbf{j} \otimes_{\mathbb{F}} \mathbf{k}$. A *left/right type DD bimodule* is a left/left type DD bimodule over $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$.

A *left/right type DA bimodule* is a $\mathbf{j} - \mathbf{k}$ bimodule equipped with maps for $i \geq 1$:

$$\delta_i^1: X \otimes_{\mathbf{k}} \overbrace{\mathcal{B} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}}^{i-1} \rightarrow \mathcal{A} \otimes_{\mathbf{j}} X,$$

satisfying the structure equation

$$\begin{aligned} 0 &= (\mu_1^{\mathcal{A}} \otimes \text{Id}_X) \circ \delta_i^1(x \otimes a_1 \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^{i-1} \delta_i^1(x \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_1^{\mathcal{B}}(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^{i-2} \delta_i^1(x \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_2^{\mathcal{B}}(a_j \otimes a_{j+1}) \otimes a_{j+2} \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^i (\mu_2^{\mathcal{A}} \otimes \text{Id}_X) \circ (\text{Id}_{\mathcal{A}} \otimes \delta_{i-j+1}^1) \circ (\delta_j^1(x \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes a_j \otimes \cdots \otimes a_{i-1}) \end{aligned}$$



Here, we think of \mathcal{B} as the algebra of inputs, and \mathcal{A} as the output algebra.

Example 2.2. Fix an algebra \mathcal{A} over \mathbf{k} . The identity bimodule ${}^{\mathcal{A}}\text{Id}_{\mathcal{A}}$ is the type DA bimodule whose underlying $\mathbf{k} - \mathbf{k}$ bimodule is \mathbf{k} , so that the maps δ_j^1 have the form

$$\delta_j^1: \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^{j-1} \rightarrow \mathcal{A}$$

and whose operations are given by $\delta_2^1(a) = a$, and $\delta_j^1 = 0$ for $j = 1$ and $j > 2$.

Example 2.3. Let $\phi: \mathcal{B} \rightarrow \mathcal{A}$ be a homomorphism of DG algebras over a base ring \mathbf{k} . This can be viewed as a bimodule ${}^{\mathcal{A}}[\phi]_{\mathcal{B}}$ with a single generator over \mathbf{k} , which we denote $\mathbf{1}$, with

$$\delta_2^1(\mathbf{1}, b) = \phi(b) \otimes \mathbf{1}.$$

The grading shifts are as follows:

$$\delta_i^1: X_{d_1; \ell_1} \otimes_{\mathbf{k}} \mathcal{B}_{d_2; \ell_2} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}_{d_i; \ell_i} \rightarrow (\mathcal{A} \otimes X)_{i-2+\sum d_j; \sum \ell_j}.$$

A left/right type AA bimodule over \mathcal{A} and \mathcal{B} is a left/right bimodule over $\mathbf{I}(\mathcal{A})$ and $\mathbf{I}(\mathcal{B})$, equipped with maps indexed by integers $i, j \geq 0$,

$$m_{i|1|j}: \overbrace{\mathcal{A} \otimes_{\mathbf{I}(\mathcal{A})} \cdots \otimes_{\mathbf{I}(\mathcal{A})} \mathcal{A}}^i \otimes X \otimes_{\mathbf{I}(\mathcal{B})} \overbrace{\mathcal{B} \otimes_{\mathbf{I}(\mathcal{B})} \cdots \otimes_{\mathbf{I}(\mathcal{B})} \mathcal{B}}^j \rightarrow X.$$

defined for all non-negative integers i and j , satisfying a structure equation which we will state shortly. The maps $m_{i|1|j}$ can be assembled to form a map

$$m: \mathcal{T}^*(\mathcal{A}) \otimes_{\mathbf{I}(\mathcal{A})} M \otimes_{\mathbf{I}(\mathcal{B})} \mathcal{T}^*(\mathcal{B}) \rightarrow M$$

that is represented by the diagram

$$\begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \swarrow \quad \downarrow \quad \searrow \\ m \\ \downarrow \end{array}$$

and satisfies the structure equation (for all $\underline{a} \in \mathcal{T}^*(\mathcal{A})$ and $\underline{b} \in \mathcal{T}^*(\mathcal{B})$):

$$\begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \swarrow \quad \downarrow \quad \searrow \\ m \end{array} + \begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \quad \downarrow \\ \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \swarrow \quad \downarrow \quad \searrow \\ m \end{array} + \begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \quad \downarrow \\ \Delta \quad \mathbf{x} \quad \Delta \\ \swarrow \quad \downarrow \quad \searrow \\ m \end{array} = 0$$

As a basic example, if \mathcal{A} is a DG algebra, we can view it as a bimodule over itself; in this case, we write ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$. The operation $m_{0|1|0}$ is the differential on the algebra, $m_{1|1|0}(a \otimes b) = a \cdot b$, and $m_{0|1|1}(b \otimes c) = b \cdot c$, for any $a, b, c \in \mathcal{A}$ (except now b is viewed as an element in the bimodule). All other operations vanish.

Sometimes, we will need to consider right/right type AA bimodules over \mathcal{A} and \mathcal{B} . These can be thought of as left/right bimodules over \mathcal{A}^{op} (the “opposite algebra”) and \mathcal{B} .

Bimodules have opposites, defined by the straightforward generalization of Equation (2.2). For example, the bimodule ${}_{\mathcal{A}}\overline{\mathcal{A}}_{\mathcal{A}}$, the opposite bimodule of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, consists of maps from $\mathcal{A} \rightarrow \mathbb{F}$; more precisely given $(d, \ell) \in \mathbb{Z} \oplus \Lambda$,

$$(2.6) \quad \overline{\mathcal{A}}_{(d, \ell)} = \text{Hom}_{\mathbb{F}}(\mathcal{A}_{(-d, -\ell)}, \mathbb{F}).$$

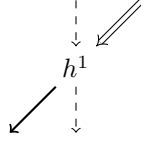
This has operations

$$\begin{aligned} m_{0|1|0}(x \mapsto \psi(x)) &= (x \mapsto \psi(dx)) \\ m_{1|1|0}(a \otimes (x \mapsto \psi(x))) &= (x \mapsto \psi(x \cdot a)) \\ m_{0|1|1}((x \mapsto \psi(x)) \otimes b) &= (x \mapsto \psi(b \cdot x)) \end{aligned}$$

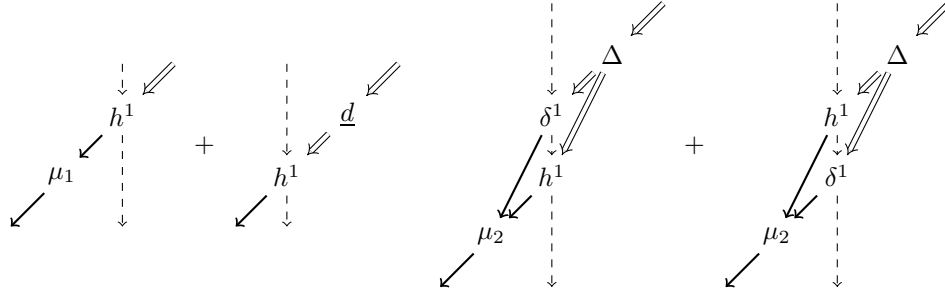
A *morphism* between type DA bimodules $h^1: {}^{\mathcal{A}}X_{\mathcal{B}} \rightarrow {}^{\mathcal{A}}Y_{\mathcal{B}}$ is a sequence of maps

$$\{h_j^1: X \otimes_{\mathbf{k}} \overbrace{\mathcal{B} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}}^{j-1} \rightarrow \mathcal{A} \otimes_{\mathbf{j}} Y\}_{j=1}^{\infty},$$

abbreviated



The differential of h^1 is the morphism dh^1 represented as the sum

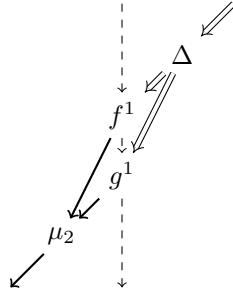


A *homomorphism* is a morphism whose differential is zero. Two homomorphisms are *homotopic* if their difference is the differential of another morphism.

Morphisms can be composed; given morphisms

$$f^1: {}^{\mathcal{A}}X_{\mathcal{B}} \rightarrow {}^{\mathcal{A}}Y_{\mathcal{B}} \quad g^1: {}^{\mathcal{A}}Y_{\mathcal{B}} \rightarrow {}^{\mathcal{A}}Z_{\mathcal{B}},$$

the composition is defined by the picture



Morphisms, homomorphisms, and homotopies can be defined for bimodules of other types in a straightforward way; see [11, Section 2.2.4].

2.7. Tensor products of bimodules. We recall here the tensor products of various types of bimodules; see [11, Section 2.3.2] for details. Let \mathcal{A} , \mathcal{B} , \mathcal{C} be three differential graded algebras over base rings \mathbf{i} , \mathbf{j} , and \mathbf{k} respectively, and let ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y_{\mathcal{C}}$ be type DA bimodules, their tensor product is a type DA bimodule structure on the vector space $X \otimes Y$ (where the tensor product is taken over the ground ring of \mathcal{B}), with structure maps $\delta^1: X \otimes_{\mathbf{j}} Y \rightarrow \mathcal{A} \otimes_{\mathbf{i}} X \otimes_{\mathbf{j}} Y$ which can be represented as

$$\begin{array}{ccc} & & \swarrow \\ & \downarrow & \delta_Y \\ & \downarrow & \\ \delta_X^1 & \swarrow & \\ & \downarrow & \end{array}$$

Here the map δ_Y is obtained by iterating δ_Y^1 (as in Equation (2.5)).

The following is immediate from the definition:

Lemma 2.4. *Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be four differential graded algebras, and fix type DA bimodules ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y_{\mathcal{C}}$ and ${}^{\mathcal{C}}Z_{\mathcal{D}}$. Then, there is an isomorphism*

$$({}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y_{\mathcal{C}}) \boxtimes {}^{\mathcal{C}}Z_{\mathcal{D}} \cong {}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes ({}^{\mathcal{B}}Y_{\mathcal{C}} \boxtimes {}^{\mathcal{C}}Z_{\mathcal{D}}).$$

□

Given a type DA bimodule ${}^{\mathcal{A}}X_{\mathcal{B}}$ and a type DD bimodule ${}^{\mathcal{B}}Y_{\mathcal{C}}$, their tensor product ${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y_{\mathcal{C}}$, when it makes sense, is a type DD bimodule over \mathcal{A} and \mathcal{C} . The type DD structure map is described by

$$(2.7) \quad \begin{array}{ccc} & & \downarrow \\ & \downarrow & \delta_Y \\ & \downarrow & \\ \delta_X^1 & \swarrow & \Pi \\ & \downarrow & \end{array}$$

where $\Pi(b_1 \otimes \cdots \otimes b_j) = b_1 \cdots b_j$.

Of course, the sum implicit in the above description is not always finite; we describe a case where it is. (See also [11, Section 2.2.4].) Consider the map

$$\delta_Y^j: Y \rightarrow (\mathcal{B}^{\otimes j}) \otimes Y \otimes (\mathcal{C}^{\otimes j})$$

obtained by iterating δ^1 (i.e. so that the map δ_Y appearing in Equation (2.7) is given as $\delta_Y = \sum_{j=0}^{\infty} \delta_Y^j$).

Definition 2.5. *For fixed integer $j \geq 1$, a length j \mathcal{B} -sequence out of $\mathbf{y} \in X$ is any sequence of algebra elements (b_1, \dots, b_j) in \mathcal{B} with the property that, for a suitable*

choice of $\mathbf{z} \in X$ and sequence (c_1, \dots, c_j) in \mathcal{C} , $(b_1 \otimes \dots \otimes b_j) \otimes \mathbf{z} \otimes (c_1 \otimes \dots \otimes c_j)$ appears with non-zero multiplicity in $\delta^j(\mathbf{y})$.

Definition 2.6. Fix DG algebras \mathcal{A} , \mathcal{B} , and \mathcal{C} , and bimodules ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y^{\mathcal{C}}$. We say that X is Y -compatible over \mathcal{B} , or when it is unambiguous, simply Y -compatible if for any non-zero $\mathbf{x} \otimes \mathbf{y} \in X \otimes Y$, if j is a sufficiently large integer, then for any length j \mathcal{B} -sequence out of \mathbf{y} (b_1, \dots, b_j) , we have that $\delta_{j+1}^1(\mathbf{x}, b_1, \dots, b_j) = 0$. Similarly, a morphism $\phi \in \text{Mor}(X, X')$ is called Y -compatible if for any $\mathbf{x} \otimes \mathbf{y} \in X \otimes Y$, if j is sufficiently large, then for any length j \mathcal{B} -sequence out of \mathbf{y} (b_1, \dots, b_j) , we have that $\phi_{j+1}^1(\mathbf{x}, b_1, \dots, b_j) = 0$. If X and X' are Y -compatible, we say that they are Y -compatibly homotopy equivalent if there are Y -compatible morphisms $\phi: X \rightarrow X'$, $\psi: X' \rightarrow X$, $h: X \rightarrow X$, and $h': X' \rightarrow X'$, so that $d\phi = 0$, $d\psi = 0$, $\psi \circ \phi = \text{Id}_X + dh$ and $\phi \circ \psi = \text{Id}_{X'} + dh'$.

Proposition 2.7. Fix ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y^{\mathcal{C}}$. If X is Y -compatible, we can form the type DD bimodule $X \boxtimes Y$, as defined in Equation (2.7). Moreover, if ${}^{\mathcal{A}}X'_{\mathcal{B}}$ is also Y -compatible, and it is Y -compatibly homotopy-equivalent to X , then $X \boxtimes Y$ and $X' \boxtimes Y$ are homotopy equivalent type DD bimodules.

Proof. It is straightforward to see that the Y -compatibility ensures that all the infinite sums appearing in the needed maps are all finite. \square

For a very simple special case, suppose that ${}^{\mathcal{A}}X_{\mathcal{B}}$ has the property that for all sufficiently large j , $\delta_j^1 = 0$. This is Y -compatible for any type DD structure ${}^{\mathcal{B}}Y^{\mathcal{C}}$.

Lemma 2.8. Let \mathcal{A} and \mathcal{B} be DG modules, and let ${}^{\mathcal{A}}M_{\mathcal{B}}$ be a type DA bimodule with the property that $\delta_j^1 = 0$ for all sufficiently large j . Then, M is homotopy equivalent to a type DA bimodule with $\delta_j^1 = 0$ for $j > 2$.

Proof. The DA bimodule M' is the bar resolution of M ; see [7] or [11]. Explicitly,

$${}^{\mathcal{A}}M'_{\mathcal{B}} = {}^{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\text{Bar}^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$$

and the homotopy equivalence is obtained from the homotopy equivalence

$${}^{\mathcal{B}}\text{Id}_{\mathcal{B}} \rightarrow {}^{\mathcal{B}}\text{Bar}^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$$

by tensor product with the identity map on M . (Boundedness of M ensures that this is a homotopy equivalence; compare [11, Lemma 2.3.19].) \square

We have the following version of associativity (compare [11, Proposition 2.3.15]):

Lemma 2.9. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be four differential graded algebras; ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}_C Z^{\mathcal{D}}$ are bimodules of type DA ; and, ${}^{\mathcal{B}}Y^{\mathcal{C}}$ is of type DD . Suppose moreover that ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}_C Z^{\mathcal{D}}$ are both bounded, in the sense that for sufficiently large j , $\delta_j^1 = 0$. Then,

$$({}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y^{\mathcal{C}}) \boxtimes {}_C Z^{\mathcal{D}} \simeq {}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes ({}^{\mathcal{B}}Y^{\mathcal{C}} \boxtimes {}_C Z^{\mathcal{D}}).$$

Proof. This is clear if $\delta_j^1 = 0$ for all $j > 2$ on ${}^{\mathcal{A}}X_{\mathcal{B}}$. We can reduce to this case by Lemma 2.8. \square

2.8. Taking homology of bimodules. By the “homological perturbation lemma”, \mathcal{A}_∞ structures persist after taking homology. (See for example [7] for the case of algebras.) We will make use of the analogous result for bimodules.

We start with a standard result from homological algebra:

Lemma 2.10. *Let $\mathbf{k} \cong \mathbb{F}^k$ for some k . Let Y be an S -graded chain complex over \mathbf{k} , and $Z = H(Y)$. Then, Z is also an S -graded chain complex (with trivial differential), and there are S -graded chain maps $f: Z \rightarrow Y$ and $g: Y \rightarrow Z$ and an S -graded map $T: Y \rightarrow Y$ with $T \circ T = 0$ and $g \circ f = \text{Id}_Z$, $f \circ g = \text{Id}_Y + \partial \circ T + T \circ \partial$.*

We will use two versions of the homological perturbation lemma:

Lemma 2.11. *Let ${}_{\mathcal{A}}Y_{\mathcal{B}}$ be a strictly unital \mathcal{A}_∞ bimodule with grading set S , let Z denote its homology. Let $f: Z \rightarrow Y$ be the homotopy equivalence of S -graded complexes of \mathbf{k} -modules from Lemma 2.10. Then there is an \mathcal{A}_∞ bimodule structure on Z , denoted ${}_{\mathcal{A}}Z_{\mathcal{B}}$, and an \mathcal{A}_∞ homotopy equivalence $\phi: {}_{\mathcal{A}}Z_{\mathcal{B}} \rightarrow {}_{\mathcal{A}}Y_{\mathcal{B}}$ with $\phi_{0|1|0} = f$.*

Proof. By hypothesis, we have maps $f: Z \rightarrow Y$ and $g: Y \rightarrow Z$ so that $f \circ g = \text{Id} + \partial \circ T + T \circ \partial$. The differential on Z vanishes; i.e. $m_{0|1|0} = 0$. Operations $m_{i|1|j}$ with $i + j > 0$ are described by

$$m^Z(\underline{a} \otimes \mathbf{x} \otimes \underline{b}) = \underline{a} \quad \mathbf{x} \quad \underline{b} + \underline{a} \quad \mathbf{x} \quad \underline{b} + \underline{a} \quad \mathbf{x} \quad \underline{b} + \cdots$$

where $a_1 \otimes \cdots \otimes a_i \in \underline{a} \in \mathcal{T}^*(\mathcal{A})$ and $b_1 \otimes \cdots \otimes b_j \in \underline{b} \in \mathcal{T}^*(\mathcal{B})$; and each node labelled by m contains an operation $m_{a|1|b}$ with $a + b > 0$; where the splitting operator Δ is generalized to have arbitrarily many outputs. With those operations, we have an \mathcal{A}_∞ -bimodule homomorphism defined by $\phi_{0|1|0} = f$ and $\phi_{i|1|j}$ with

$$\phi(\underline{a} \otimes \mathbf{x} \otimes \underline{b}) = \begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \quad \downarrow \\ f \\ \downarrow \\ m \\ \downarrow \\ T \\ \downarrow \end{array} + \begin{array}{c} \underline{a} \quad \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \quad \downarrow \\ \Delta \quad f \quad \Delta \\ \downarrow \quad \downarrow \quad \downarrow \\ m \\ \downarrow \\ T \\ \downarrow \end{array} + \dots$$

We will also use a variant for DA bimodules:

Lemma 2.12. *Let ${}^{\mathcal{A}}Y_{\mathcal{B}}$ be a strictly unital type DA bimodule with grading set S , and let ${}^{\mathcal{A}}Z$ be type D structure over \mathcal{A} . Suppose that there are type D structure homomorphisms $f: {}^{\mathcal{A}}Z \rightarrow {}^{\mathcal{A}}Y$ (i.e. as the notation suggests, we are forgetting here about the right \mathcal{B} -action) and $g: {}^{\mathcal{A}}Y \rightarrow {}^{\mathcal{A}}Z$ and a type D structure morphism $T: {}^{\mathcal{A}}Y \rightarrow {}^{\mathcal{A}}Y$ so that*

$$f \circ g = \text{Id}_Z, \quad g \circ f = \text{Id}_Y + dT, \quad T \circ T = 0.$$

(Here, $T \circ T$ the composite of type D structures; see Equation (2.4).) Then ${}^A Z$ can be turned into a strictly unital type DA bimodule, denoted ${}^A Z_B$; and there is an A_∞ homotopy equivalence $\phi: {}^A Z_B \rightarrow {}^A Y_B$ with $\phi_1^1 = f$.

Proof. ${}^{\mathcal{A}}Z$ is already equipped with an action δ_1^1 . For $j > 1$, operations δ_j^1 on $\underline{b} = b_1 \otimes \cdots \otimes b_{j-1}$ are specified by

$$\delta^1(\mathbf{x} \otimes \underline{b}) = \begin{array}{c} \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \\ f \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ g \quad \downarrow \\ \downarrow \quad \downarrow \\ \Pi \quad \downarrow \end{array} + \begin{array}{c} \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \\ f \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ g \quad \downarrow \\ \downarrow \quad \downarrow \\ \Pi \quad \downarrow \end{array} + \begin{array}{c} \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \\ f \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ g \quad \downarrow \\ \downarrow \quad \downarrow \\ \Pi \quad \downarrow \end{array} + \cdots$$

similarly, define $\phi_1^1 = f$, and for $j > 1$, ϕ^1 on $\underline{b} = b_1 \otimes \cdots \otimes b_{j-1}$ is given by

$$\phi^1(\mathbf{x} \otimes \underline{b}) = \begin{array}{c} \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \\ f \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ \Pi \quad \downarrow \end{array} + \begin{array}{c} \mathbf{x} \quad \underline{b} \\ \downarrow \quad \downarrow \\ f \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ m \quad \downarrow \\ \downarrow \quad \downarrow \\ T \quad \downarrow \\ \downarrow \quad \downarrow \\ \Pi \quad \downarrow \end{array} + \cdots$$

□

2.9. Koszul duality. Let \mathcal{A} and \mathcal{B} be two differential graded algebras Λ . Let ${}^{\mathcal{A}}X^{\mathcal{B}}$ be a type DD bimodule and ${}_{\mathcal{B}}Y_{\mathcal{A}}$ be a type AA bimodule. We say that these two bimodules are *quasi-inverses* if there are homotopy equivalences

$$\begin{aligned} {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} &\simeq {}^{\mathcal{A}}\text{Id}_{\mathcal{A}} \\ {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} &\simeq {}_{\mathcal{B}}\text{Id}_{\mathcal{B}} \end{aligned}$$

In this case, we call ${}^{\mathcal{A}}X^{\mathcal{B}}$ resp. ${}_{\mathcal{B}}Y_{\mathcal{A}}$ *quasi-invertible type DD* resp. *type AA* bimodules.

To see that X and Y are quasi-inverses, it suffices to exhibit homomorphisms

$$\begin{aligned}\phi^1: {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} &\rightarrow {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}} \\ \psi^1: {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} &\rightarrow {}_{\mathcal{B}}\mathrm{Id}_{\mathcal{B}}\end{aligned}$$

so that the maps

$$\begin{aligned}\mathrm{Id}_{\mathcal{A}} \boxtimes \phi^1: {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} &\rightarrow {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \\ \psi^1 \boxtimes \mathrm{Id}_{\mathcal{B}}: {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} &\rightarrow {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}\end{aligned}$$

induce isomorphisms on homology, according to [11, Corollary 2.4.4].

Definition 2.13. Let \mathbf{k} be a ring that is a finite direct sum of \mathbb{F} . Let \mathcal{A} and \mathcal{B} be two DG algebras, both of which are positively graded over \mathbf{k} (in the sense of Definition 2.1). A bimodule ${}^{\mathcal{A}}X^{\mathcal{B}}$ is called a Koszul dualizing bimodule if it satisfies the following properties:

- (K-1) ${}^{\mathcal{A}}X^{\mathcal{B}}$ is graded by Λ .
- (K-2) ${}^{\mathcal{A}}X^{\mathcal{B}}$ has rank one; i.e. there is an isomorphism

$${}^{\mathcal{A}}X^{\mathcal{B}} \cong \mathbf{k}$$

as \mathbf{k} - \mathbf{k} bimodules.

- (K-3) The image of δ^1 is contained in $\bigoplus_{\lambda \in \Lambda \setminus 0} {}_{\mathcal{A}}\lambda \otimes {}^{\mathcal{A}}X^{\mathcal{B}} \otimes \lambda_{\mathcal{B}}$.
- (K-4) ${}^{\mathcal{A}}X^{\mathcal{B}}$ is quasi-invertible.

If such a bimodule exists, then one says that \mathcal{A} and \mathcal{B} are Koszul dual to one another.

Definition 2.14. If ${}^{\mathcal{A}}X^{\mathcal{B}}$ is a type DD bimodule, we can form the candidate quasi-inverse module

$$(2.8) \quad {}^{\mathcal{A}}\mathrm{Mor}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}) \cong {}_{\mathcal{B}}\overline{\mathcal{B}}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

Lemma 2.15. For a type DD bimodule satisfying properties (K-2) and (K-3), the candidate quasi-inverse module has a subcomplex $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathbf{k}$ which is isomorphic to \mathbf{k} .

Proof. For any DD bimodule, $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathcal{A}$ is a subcomplex of the candidate quasi-inverse module. Condition (K-3) ensures that the induced differential on $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathcal{A}$ is simply the differential on \mathcal{A} ; and hence $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathbf{k}$ is a subcomplex. Condition (K-2) now gives the desired identification of the subcomplex with \mathbf{k} . \square

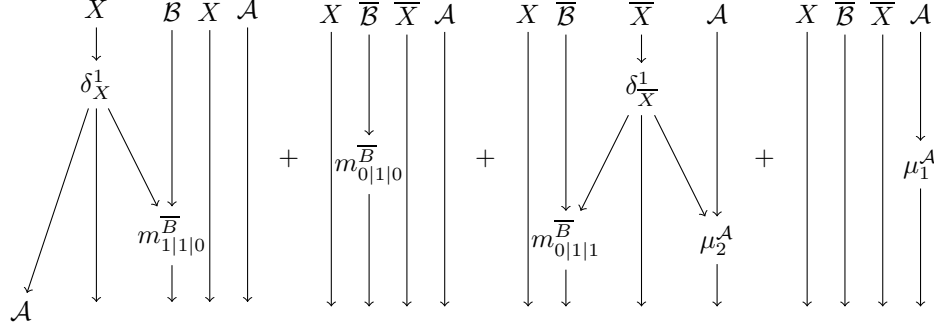
Lemma 2.16. Fix \mathbf{k} as above in Definition 2.13, and let ${}^{\mathcal{A}}X^{\mathcal{B}}$ be a type DD bimodule satisfying Conditions (K-2) and (K-3). Suppose that the inclusion map from \mathbf{k} to the candidate quasi-inverse module coming from Lemma 2.15 induces an isomorphism on homology. Then the candidate quasi-inverse module is a quasi-inverse of ${}^{\mathcal{A}}X^{\mathcal{B}}$.

Proof. Let ${}_{\mathcal{B}}Y_{\mathcal{A}}$ be the candidate quasi-inverse. Our goal is to show, under the stated hypotheses, that ${}_{\mathcal{B}}Y_{\mathcal{A}}$ is a quasi-inverse to ${}^{\mathcal{A}}X^{\mathcal{B}}$.

To this end, consider a DA bimodule by tensoring X with Y :

$${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} = {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\overline{\mathcal{B}}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

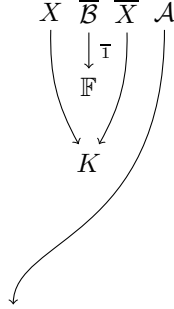
The differential on this bimodule is given pictorially by:



There is a natural map $h: {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \rightarrow {}^{\mathcal{A}}\text{Id}_{\mathcal{A}}$, defined by $x \otimes \psi \rightarrow \psi(x \otimes \mathbf{1})$, where here $\mathbf{1} \in \mathcal{B}$ is the unit; i.e. given $\mathbf{x} \in X$ and $\psi \in Y = {}^{\mathcal{A}}\text{Mor}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}, {}^{\mathcal{A}}\text{Id}_{\mathcal{A}})$, $x \otimes \mathbf{1} \in {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}$, so $\psi(x \otimes \mathbf{1}) \in {}^{\mathcal{A}}\text{Id}_{\mathcal{A}}$. This natural map can be viewed as a DA morphism. Recall that a DA morphism ${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \rightarrow {}^{\mathcal{A}}\text{Id}_{\mathcal{A}}$ is specified by a sequence of maps indexed by $j \geq 1$:

$$h_j^1: {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{j-1} \rightarrow \mathcal{A}.$$

The morphism under question has $h_j^1 = 0$ for $j > 1$, and it was specified above in the case where $j = 1$. Pictorially, h is represented as follows:



Where here K denotes the Kronecker pairing, and $\bar{1}$ is dual to the map $1: \mathbb{F} \rightarrow \mathcal{B}$.

The verification that h is a DA bimodule homomorphism is straightforward.

So far, we have not used the hypothesis on the homology of ${}_{\mathcal{B}}Y_{\mathcal{A}}$ (which is needed to verify that h is a homotopy equivalence). From the hypothesis, there is a rank one bimodule ${}_{\mathcal{B}}Z_{\mathcal{A}}$; this is the bimodule structure induced on \mathbf{k} from its quasi-isomorphism with ${}_{\mathcal{B}}Y_{\mathcal{A}}$ as in Lemma 2.11. Let

$$\phi: {}_{\mathcal{B}}Z_{\mathcal{A}} \rightarrow {}_{\mathcal{B}}Y_{\mathcal{A}}$$

be the induced bimodule quasi-isomorphism. In more detail, denote the isomorphism $\mathbf{k} \cong X$ by $\iota \mapsto x_{\iota}$; taking its dual and using an isomorphism $\mathbf{k} \cong \bar{\mathbf{k}}$ induced from an \mathbb{F} -basis for \mathbf{k} , we get an isomorphism $\mathbf{k} \rightarrow \bar{X}$ denoted $\iota \mapsto \bar{x}_{\iota}$. Now,

$$\phi_{0|1|0}(\iota) = \bar{1} \otimes \bar{x}_{\iota} \otimes 1.$$

The map ϕ satisfies the following two properties:

- (1) On the bimodule ${}_B Z_{\mathcal{A}}$, for all $k > 0$,
- (2.9) $m_{k|1|0} = 0$
- (2) On the bimodule homomorphism, for all $k > 0$,
- (2.10) $\phi_{k|1|0} = 0$.

Both of these follow from the formulas from Lemma 2.11. In more detail, let $\Delta \subset \Lambda \times \Lambda$ denote the diagonal; i.e. the elements of the form (λ, λ) . The image of $m_{k|1|0}(a_1, \dots, a_{k-1}, x)$ would be supported in the portion in grading $(\lambda(x) + (\lambda(a_1) + \dots + \lambda(a_{k-1}), 0)) \cap \Delta$, which, since $\lambda(x) \in \Delta$, means that $\lambda(a_1) + \dots + \lambda(a_{k-1}) = 0$. By positivity of the grading on \mathcal{A} , we conclude that each $\lambda(a_i) = 0$, and hence that each $a_i \in \mathbf{k}$. But now positivity (Definition 2.1) implies Equation (2.9). Equation (2.10) follows from the same reasoning.

To verify h is a DA bimodule quasi-isomorphism, consider the induced AA -bimodule homomorphism

$$(\text{Id}_{\mathcal{A}} \boxtimes h): {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B Y_B = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B \overline{\mathcal{B}}_B \boxtimes {}_B X^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \rightarrow {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

To verify that it is a quasi-isomorphism, we form the following composite

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B Z_{\mathcal{A}} \xrightarrow{\text{Id}_{\mathcal{A}} \boxtimes \kappa} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B \overline{\mathcal{B}}_B \boxtimes {}_B X^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \xrightarrow{\text{Id}_{\mathcal{A}} \boxtimes h} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}},$$

which we denote κ . From Equation (2.9), we conclude that the chain complex on the left is identified with \mathcal{A} . From Equation (2.10), we conclude that that $\kappa_{0|1|0}$ is the chain map $(\text{Id}_{\mathcal{A}} \otimes h_1^1) \circ (\text{Id}_{\mathcal{A}} \boxtimes X \otimes \phi_{0|1|0})$ that sends $a \otimes x_{\iota} \otimes \iota$ to a ; in particular, it is an isomorphism of chain complexes. \square

Remark 2.17. *In fact, if ${}^{\mathcal{A}}X^{\mathcal{B}}$ is quasi-invertible, the quasi-inverse is always given by the above bimodules by the argument from [11, Proposition 9.2]; i.e. there are quasi-isomorphisms:*

$$\begin{aligned} {}_B Y_{\mathcal{A}} &\simeq \text{Mor}_B({}_B \mathcal{B}_B, {}_B Y_{\mathcal{A}}) \\ &\simeq \text{Mor}^{\mathcal{A}}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B \mathcal{B}_B, {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B \mathcal{B}_B) \\ &\simeq \text{Mor}^{\mathcal{A}}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_B \mathcal{B}_B, {}^{\mathcal{A}}\text{Id}_{\mathcal{A}}); \end{aligned}$$

the first of these is true for arbitrary ${}_B Y_{\mathcal{A}}$, the second uses the fact that $X \boxtimes$ induces an equivalence of categories, and the third uses the fact that X and Y are quasi-inverses.

3. THE ALGEBRAS

We describe differential graded algebras used in the construction of our knot invariant. In fact, we will find it convenient to work with a more general construction $\mathcal{B}(m, k, \mathcal{S})$, where $0 \leq k \leq m + 1$, and \mathcal{S} is an arbitrary subset of $\{1, \dots, m\}$. The integer m is called the *index*. The integer k is the *number of occupied positions*; together with the index, it determines the base ring $\mathbf{k} = \mathbf{I}(\mathcal{B})$. When $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \{1, \dots, m\}$, then $\mathcal{B}(m, k, \mathcal{S}_1)$ will be a differential subalgebra of $\mathcal{B}(m, k, \mathcal{S}_2)$.

When $\mathcal{S} = \emptyset$, the differential on the algebra $\mathcal{B}(m, k, \emptyset) = \mathcal{B}(m, k)$ vanishes. In fact, this algebra will be defined as the quotient of a larger algebra $\mathcal{B}_0(m, k)$, which we define first.

When constructing the knot invariant for a knot with bridge number n , we will have $m = 2n$, $k = n$, and \mathcal{S} will correspond to those (n) strands that are oriented upwards. In fact, for the purposes of the knot invariant, it would suffice to work in a summand corresponding to certain idempotents; see Remark 11.13. We have chosen to use the larger algebra in our constructions, as it satisfies a duality described in Section 3.8.

3.1. The algebra $\mathcal{B}_0(m, k)$. We define $\mathcal{B}_0(m, k)$, which is a graded algebra over $\mathbb{F}[U_1, \dots, U_m]$, whose (Alexander multi-grading) set is $(\frac{1}{2}\mathbb{Z})^m$. (See Section 2.2.) Basic idempotents in $\mathcal{B}_0(m, k)$ correspond to *idempotent states*, or *I-states* for short, $\mathbf{x} = (x_1, \dots, x_k)$, which are increasing sequences of integers

$$0 \leq x_1 < \dots < x_k \leq m.$$

The basic idempotent corresponding to \mathbf{x} will be denoted by $\mathbf{I}_{\mathbf{x}}$. The elements $\mathbf{I}_{\mathbf{x}}$ are generators of a ring $\mathbf{k} = \mathbf{I}(m, k)$ satisfying:

$$\mathbf{I}_{\mathbf{x}} \cdot \mathbf{I}_{\mathbf{y}} = \begin{cases} \mathbf{I}_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{y} \\ 0 & \text{if } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

Remark 3.1. Recall from Section 1 that these idempotents can be interpreted as marking intervals which are intersections of the regions in a knot diagram with the $y = t$ slice.

The unit $\mathbf{1}$ in the algebra is given by the sum of the basic idempotents.

Given an I-state \mathbf{x} , define its *weight* $v^{\mathbf{x}} \in \mathbb{Z}^m$ by

$$(3.1) \quad v_i^{\mathbf{x}} = \#\{x \in \mathbf{x} \mid x \geq i\}.$$

Given two I-states \mathbf{x} and \mathbf{y} , define their *minimal relative weight vector* $w^{\mathbf{x}, \mathbf{y}} \in (\frac{1}{2}\mathbb{Z})^m$ to be given by

$$w_i^{\mathbf{x}, \mathbf{y}} = \frac{1}{2} |v_i^{\mathbf{x}} - v_i^{\mathbf{y}}|.$$

$\mathcal{B}_0(m, k)$ is defined so that there is an identification of $\mathbb{F}[U_1, \dots, U_m]$ -modules $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}} \cong \mathbb{F}[U_1, \dots, U_m]$; denote the identification by

$$\phi^{\mathbf{x}, \mathbf{y}}: \mathbb{F}[U_1, \dots, U_m] \rightarrow \mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}}$$

A grading by $(\frac{1}{2}\mathbb{Z})^m$ on $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}}$ is specified by

$$(3.2) \quad w(\phi(U_1^{t_1} \dots U_m^{t_m})) = w^{\mathbf{x}, \mathbf{y}} + (t_1, \dots, t_m),$$

for non-negative integers t_1, \dots, t_m .

Multiplication

$$\left(\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}} \right) * \left(\mathbf{I}_{\mathbf{y}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{z}} \right) \rightarrow \left(\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{z}} \right)$$

is the unique non-trivial, grading-preserving $\mathbb{F}[U_1, \dots, U_m]$ -equivariant map. Explicitly, given I-states \mathbf{x} , \mathbf{y} , and \mathbf{z} , if we define

$$g^{\mathbf{x}, \mathbf{y}, \mathbf{z}} = U^{t_1} \dots U^{t_m},$$

where

$$t_i = w_i^{\mathbf{x}, \mathbf{y}} + w_i^{\mathbf{y}, \mathbf{z}} - w_i^{\mathbf{x}, \mathbf{z}},$$

then for $a, b \in \mathbb{F}[U_1, \dots, U_m]$,

$$\phi^{\mathbf{x}, \mathbf{y}}(a) * \phi^{\mathbf{y}, \mathbf{z}}(b) = \phi^{\mathbf{x}, \mathbf{z}}(a \cdot b \cdot g^{\mathbf{x}, \mathbf{y}, \mathbf{z}}).$$

Suppose that \mathbf{x} is an I-state with $j-1 \in \mathbf{x}$ but $j \notin \mathbf{x}$. Then, we can form a new I-state $\mathbf{y} = \mathbf{x} \cup \{j\} \setminus \{j-1\}$, and let $R_j^{\mathbf{x}} = \phi^{\mathbf{x}, \mathbf{y}}(1)$. We define

$$R_j = \sum_{\{\mathbf{x} \mid j-1 \in \mathbf{x}, j \notin \mathbf{x}\}} R_j^{\mathbf{x}},$$

so that

$$\mathbf{I}_{\mathbf{x}} \cdot R_j = \begin{cases} R_j^{\mathbf{x}} & \text{if } j-1 \in \mathbf{x} \text{ and } j \notin \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, let $L_j^{\mathbf{y}} = \phi^{\mathbf{y}, \mathbf{x}}(1)$, and

$$L_j = \sum_{\{\mathbf{y} \mid j \in \mathbf{y}, j-1 \notin \mathbf{y}\}} L_j^{\mathbf{y}}.$$

Less formally R_j moves one of the coordinates of an I-state from the $(j-1)^{st}$ position to the j^{th} , and L_j changes it back from j^{th} to $(j-1)^{st}$.

The elements L_j and R_j are called the *left shifts* and *right shifts* respectively. See Figure 5 for a picture.

For $i = 1, \dots, m$, each U_i induces a corresponding algebra element in \mathcal{B}_0 , which is given as $\sum_{\mathbf{x}} \phi^{\mathbf{x}, \mathbf{x}}(U_i)$. For notational simplicity, we also denote this induced element by U_i .

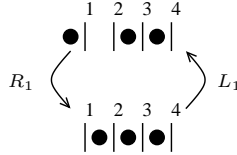


FIGURE 5. **Picture of algebra elements in $\mathcal{B}_0(4, 3)$.** Idempotents $\mathbf{I}_{\{0,2,3\}}$ and $\mathbf{I}_{\{1,2,3\}}$ are pictured here; the algebra elements L_1 and R_1 which connect them are indicated (e.g. $\mathbf{I}_{\{0,2,3\}} \cdot R_1 \cdot \mathbf{I}_{\{1,2,3\}}$ is non-zero). Here, $\mathbf{I}_{\{0,2,3\}} \cdot R_1 \cdot L_1 = \mathbf{I}_{\{0,2,3\}} \cdot U_1$.

Proposition 3.2. *The weight function from Equation (3.2) descends to a grading $w = (w_1, \dots, w_m)$ on $\mathcal{B}_0(m, k)$ with values in $(\frac{1}{2}\mathbb{Z})^m$.*

Proof. It suffices to show that

$$w(\phi^{\mathbf{x}, \mathbf{y}}(1) * \phi^{\mathbf{y}, \mathbf{z}}(1)) = w(\phi^{\mathbf{x}, \mathbf{y}}(1)) + w(\phi^{\mathbf{y}, \mathbf{z}}(1)).$$

which follows from the definition of $g^{\mathbf{x}, \mathbf{y}, \mathbf{z}}$. \square

We have no further need to distinguish the product on the algebra $\mathcal{B}_0(m, k)$ from other products; so we will abbreviate $a * b$ by $a \cdot b$.

In the notation from Section 2.2, $\mathcal{B}_0(m, k)$ is graded by the Abelian group $\Lambda = (\frac{1}{2}\mathbb{Z})^m$. Recall that an element $a \in \mathcal{B}_0(m, k)$ that is supported in some fixed grading $\lambda \in \Lambda$ is called *homogeneous*.

Proposition 3.3. *The algebra $\mathcal{B}_0(m, k)$ is generated over \mathbb{F} by the elements L_i , R_i , U_i , and the idempotents $\mathbf{I}_{\mathbf{x}}$.*

Proof. Let a be a homogeneous algebra elements with $\mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{y}} = a$. If $\mathbf{x} = \mathbf{y}$, then a factors as a product of U_i . Otherwise, suppose that some $i = x_t > y_t$; and indeed choose t minimal with this property. Then, it is easy to see that we can factor $a = L_i \cdot b$. Otherwise, there is some $y_t > x_t = i$, and we can choose t maximal with this property. Then, we can factor $a = R_{i+1} \cdot b$. In both cases, the total weight of b is smaller than that of a , so the result follows by induction on weight. \square

3.2. The $\mathcal{B}(m, k)$. The algebra $\mathcal{B}(m, k)$ is the quotient of $\mathcal{B}_0(m, k)$ by the relations

$$(3.3) \quad L_{i+1} \cdot L_i = 0$$

$$(3.4) \quad R_i \cdot R_{i+1} = 0;$$

and also, if $\{x_1, \dots, x_k\} \cap \{j-1, j\} = \emptyset$, then

$$(3.5) \quad \mathbf{I}_{\mathbf{x}} \cdot U_j = 0.$$

The quotient of $\mathcal{B}_0(m, k)$ by these further relations (Equation (3.3), (3.4), and (3.5)) gives the algebra $\mathcal{B}(m, k)$.

As we shall see, the relations Equations (3.3) and (3.4) guarantee that algebra elements in $\mathcal{B}(m, k)$ cannot move coordinates in the idempotents states by more than one unit.

We can define the algebra more precisely, as follows:

Definition 3.4. *Let \mathcal{J} be the (two-sided) ideal in $\mathcal{B}_0(m, k)$ that is generated by $L_{i+1} \cdot L_i$, $R_i \cdot R_{i+1}$, and $\mathbf{I}_{\mathbf{x}} \cdot U_j$, when $\{x_1, \dots, x_k\} \cap \{j-1, j\} = \emptyset$. Let $\mathcal{B}(m, k)$ be the quotient of $\mathcal{B}_0(m, k)$ by this two-sided ideal.*

Note that ideal is generated by elements that are homogeneous with respect to the weights. Thus, the grading specified by weights descends to the quotient algebra $\mathcal{B}(m, k)$.

The ideal \mathcal{J} can be understood concretely. To do so, we set up some notation.

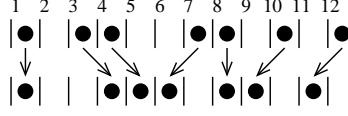


FIGURE 6. The ideal $\mathcal{I}(\mathbf{x}, \mathbf{y})$ is generated by U_1U_2 , U_3 , U_6 , U_8U_9 , and U_{11} .

Definition 3.5. Two I-states \mathbf{x} and \mathbf{y} are said to be far if there is some $i = 1, \dots, k$ with $|x_i - y_i| > 1$; otherwise they are called close enough.

Given two I-states \mathbf{x} and \mathbf{y} , we define an ideal $\mathcal{I}(\mathbf{x}, \mathbf{y}) \subset \mathbb{F}[U_1, \dots, U_m]$. If the I-states \mathbf{x} and \mathbf{y} are far, let $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathbb{F}[U_1, \dots, U_m]$. If \mathbf{x} and \mathbf{y} are close enough, let $\mathcal{I}(\mathbf{x}, \mathbf{y})$ be the ideal generated by monomials $U_{i+1} \cdots U_j$, taken over all $0 \leq i < j \leq m$ where i and j satisfy the following conditions:

- $i, j \in \{0, \dots, m\} \setminus (\mathbf{x} \cap \mathbf{y})$
- for all integers t with $i < t < j$, $t \in \mathbf{x} \cap \mathbf{y}$.
- $w_t^{\mathbf{x}, \mathbf{y}} = 0$ for all $i + 1 \leq t \leq j$.

For $i < j$ as above, we call the interval $[i + 1, j]$ a *generating interval* for \mathbf{x} and \mathbf{y} . (Observe that a generating interval can have $i + 1 = j$, and this corresponds to U_j .)

Proposition 3.6. For all I-states \mathbf{x} and \mathbf{y} ,

$$\mathbf{I}_{\mathbf{x}} \cdot \mathcal{J} \cdot \mathbf{I}_{\mathbf{y}} = \phi^{\mathbf{x}, \mathbf{y}}(\mathcal{I}(\mathbf{x}, \mathbf{y})).$$

Proof. Let

$$\mathcal{I} = \bigoplus_{\mathbf{x}, \mathbf{y}} \phi^{\mathbf{x}, \mathbf{y}}(\mathcal{I}(\mathbf{x}, \mathbf{y})).$$

First we prove $\mathcal{I} \subseteq \mathcal{J}$; i.e. for any two I-states \mathbf{x} and \mathbf{y} we have $\phi^{\mathbf{x}, \mathbf{y}}(\mathcal{I}(\mathbf{x}, \mathbf{y})) \subset \mathcal{J}$.

If \mathbf{x} and \mathbf{y} are far, so that $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathbb{F}[U_1, \dots, U_m]$, then we claim that $\phi^{\mathbf{x}, \mathbf{y}}(1)$ can be factored as a product $a \cdot R_t R_{t+1} \cdot b$ or $a \cdot L_t L_{t-1} \cdot b$ for some t . This is obvious, for example if there is an i with $x_i > y_i + 1$, then $\phi^{\mathbf{x}, \mathbf{y}}(1)$ decomposes as $a \cdot L_t \cdot L_{t-1} \cdot b$, where $t = x_i$.

If \mathbf{x} and \mathbf{y} are not far, we show that for all generating intervals $[i + 1, j]$ for \mathbf{x} and \mathbf{y} , the algebra elements $\phi^{\mathbf{x}, \mathbf{y}}(U_{i+1} \cdots U_j)$ are in \mathcal{J} .

Consider first the case where $i + 1 = j$. Then, $\phi^{\mathbf{x}, \mathbf{y}}(U_j)$ can be factored as $a \cdot \mathbf{I}_{\mathbf{w}} \cdot U_j \cdot b$, where $j - 1, j \notin \mathbf{w}$. In general, if $i < j$ is a generating interval for \mathbf{x} and \mathbf{y} , $\phi^{\mathbf{x}, \mathbf{y}}(U_{i+1} \cdots U_j)$ has a decomposition as

$$(a \cdot L_{i+1} \cdots L_{j-1}) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_j \cdot R_{j-1} \cdots R_{i+1} \cdot b,$$

where $j - 1, j \notin \mathbf{w}$. In these cases, $\mathbf{I}_{\mathbf{w}} \cdot U_j \in \mathcal{J}$. (For example, for the I-states \mathbf{x} and \mathbf{y} in Figure 6,

$$\begin{aligned} \phi^{\mathbf{x}, \mathbf{y}}(U_3) &= \phi^{\mathbf{x}, \mathbf{y}}(1) \cdot \mathbf{I}_{\mathbf{y}} \cdot U_3 \cdot \phi^{\mathbf{y}, \mathbf{y}}(1) \\ \phi^{\mathbf{x}, \mathbf{y}}(U_{11}) &= \phi^{\mathbf{x}, \mathbf{w}}(1) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_{11} \cdot L_{12} & \text{where } \mathbf{w} = \{1, 4, 5, 6, 7, 8, 9, 12\} \\ \phi^{\mathbf{x}, \mathbf{y}}(U_1 U_2) &= \phi^{\mathbf{x}, \mathbf{w}}(1) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_2 \cdot \phi^{\mathbf{w}, \mathbf{y}}(1) & \text{where } \mathbf{w} = \{0, 3, 4, 7, 8, 10, 12\} \end{aligned}$$

This completes the proof that $\mathcal{I} \subseteq \mathcal{J}$.

To prove the containment $\mathcal{J} \subseteq \mathcal{I}$, we first prove that \mathcal{I} is an ideal; i.e. if a is an arbitrary algebra element, then $a \cdot \mathcal{I} \subseteq \mathcal{I}$; and also $\mathcal{I} \cdot a \subseteq \mathcal{I}$. In view of Proposition 3.3, it suffices to verify this in the case where $a = \mathbf{I}_{\mathbf{x}} \cdot U_i$, $\mathbf{I}_{\mathbf{x}} \cdot R_i$, or $\mathbf{I}_{\mathbf{x}} \cdot L_i$.

When $a = \mathbf{I}_{\mathbf{x}} \cdot U_i$, the result follows readily.

By symmetry, we can consider the case where $a = \mathbf{I}_{\mathbf{x}} \cdot R_i \cdot \mathbf{I}_{\mathbf{y}} \neq 0$. Let t denote the index with $x_t = i - 1$ and $y_t = i$. We wish to show that $\mathbf{I}_{\mathbf{x}} \cdot R_i \cdot \phi^{\mathbf{y}, \mathbf{z}}(\mathcal{I}(\mathbf{y}, \mathbf{z})) \subset \phi^{\mathbf{x}, \mathbf{z}}(\mathcal{I}(\mathbf{x}, \mathbf{z}))$. There are cases:

- (1) If \mathbf{x} and \mathbf{z} are far, the statement is vacuously true.
- (2) If \mathbf{y} and \mathbf{z} are far, but \mathbf{x} and \mathbf{z} are close enough, then $z_t = i - 2$ and $g^{\mathbf{x}, \mathbf{y}, \mathbf{z}} = U_i$. Furthermore we have $i \notin \mathbf{x}$, $i - 1 \notin \mathbf{z}$, and $w_i^{\mathbf{x}, \mathbf{z}} = 0$. It follows that $U_i \in \mathcal{I}(\mathbf{x}, \mathbf{z})$ and this finishes the argument in this subcase.
- (3) If \mathbf{y} and \mathbf{z} are not far, and \mathbf{x} and \mathbf{z} are not far, as well, then there are two subcases according to the value of z_t .
 - (3a) If $z_t = i$, the generating intervals for \mathbf{x}, \mathbf{z} are contained in the generating intervals for \mathbf{y}, \mathbf{z} , and the containment is clear.
 - (3b) If $z_t = i - 1$, there can be at most one generating interval for (\mathbf{y}, \mathbf{z}) which is not also a generating interval for (\mathbf{x}, \mathbf{z}) , and that is an interval which terminates in i . Since $g^{\mathbf{x}, \mathbf{y}, \mathbf{z}} = U_i$, it follows that

$$\mathbf{I}_{\mathbf{x}} \cdot R_i \cdot \phi^{\mathbf{y}, \mathbf{z}}(U_j U_{j+1} \cdots U_{i-1}) = \phi^{\mathbf{x}, \mathbf{z}}(U_j U_{j+1} \cdots U_i).$$

The containment $\phi^{\mathbf{x}, \mathbf{y}}(\mathcal{I}(\mathbf{x}, \mathbf{y})) \cdot R_i \cdot \mathbf{I}_{\mathbf{z}} \subset \phi^{\mathbf{x}, \mathbf{z}}(\mathcal{I}(\mathbf{x}, \mathbf{z}))$ follows symmetrically.

We must now verify that the defining relations for \mathcal{J} are contained in \mathcal{I} . Clearly, if $\mathbf{I}_{\mathbf{x}} \cdot R_i \cdot R_{i+1} \cdot \mathbf{I}_{\mathbf{y}}$ is non-zero in \mathcal{B}_0 , then \mathbf{x} and \mathbf{y} are too far, and so $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathbb{F}[U_1, \dots, U_m]$, proving the containment. The same argument works for $L_{i+1} \cdot L_i$. Finally, if $\mathbf{x} \cap \{i - 1, i\} = \emptyset$, then U_i is a monomial corresponding to a generating interval for (\mathbf{x}, \mathbf{x}) , so $U_i \in \mathcal{I}(\mathbf{x}, \mathbf{x})$. \square

The following lemma will be useful later.

Lemma 3.7. *Let $a \in \mathcal{B}(m, k)$ be homogeneous, and suppose that $w_i(a) \in \mathbb{Z}$. Then $a \cdot R_i \neq 0$ implies that $a \cdot U_i \neq 0$; similarly, $R_i \cdot a \neq 0$ implies that $U_i \cdot a \neq 0$.*

Proof. Suppose that $b = a \cdot R_i \neq 0$, and $b = \mathbf{I}_{\mathbf{x}} \cdot b \cdot \mathbf{I}_{\mathbf{y}}$. Since $w_i(a) \in \mathbb{Z}$ and $b \neq 0$, it follows that $w_i^{\mathbf{x}, \mathbf{y}} = \frac{1}{2}$. Thus, it follows that i is not in any of the generating intervals for \mathbf{x} and \mathbf{y} , so $b \cdot U_i \neq 0$, so $a \cdot U_i \neq 0$, as well. \square

Proposition 3.8. *A homogeneous non-zero algebra element $a = \mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{y}} \in \mathcal{B}(m, k)$ is uniquely characterized by its initial (or terminal) idempotent and its weight.*

Proof. Fix initial and terminal idempotent states \mathbf{x} and \mathbf{y} . By Proposition 3.6, $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}(m, k) \cdot \mathbf{I}_{\mathbf{y}}$ is the quotient of the polynomial algebra $\mathbb{F}[U_1, \dots, U_m]$ by an ideal which is homogeneous with respect to the weights. An element of this quotient space in turn is uniquely determined by its various weights.

Note that the weight of a modulo 1 determines $w^{\mathbf{x}, \mathbf{y}}$. It follows that \mathbf{x} and $w(a)$ determines \mathbf{y} . \square

3.3. Defining $\mathcal{B}(m, k, \mathcal{S})$. Let $\mathcal{S} \subset \{1, \dots, m\}$ be a subset. Define $\mathcal{B}(m, k, \mathcal{S})$ to be the differential graded algebra obtained by adjoining new algebra elements C_i for $i \in \mathcal{S}$ to $\mathcal{B}(m, k)$, which satisfy the following properties:

- The C_i commute with all other algebra elements in $\mathcal{B}(m, k, \mathcal{S})$.
- The square of C_i vanishes.
- The differential of C_i is U_i .

This construction can be done since the U_i are in the center of $\mathcal{B}(m, k, \mathcal{S})$ and $dU_i = 0$.

More formally, if $\mathcal{S} = \{i_1, \dots, i_n\}$, then $\mathcal{B}(m, k, \mathcal{S})$ is defined by the formula

$$\mathcal{B}(m, k, \mathcal{S}) = \frac{\mathcal{B}(m, k)[C_{i_1}, \dots, C_{i_n}]}{\{C_j^2 = 0, dC_j = U_j\}_{j \in \mathcal{S}}}.$$

The algebra $\mathcal{B}(m, k, \mathcal{S})$ is equipped with a distinguished basis as an \mathbb{F} -vector space. The basis vectors correspond to the following data:

- A pair of idempotents \mathbf{x} and \mathbf{y} that are not far,
- a monomial p in $\mathbb{F}[U_1, \dots, U_m]$ that is not divisible by any monomial associated to any generating interval for \mathbf{x} and \mathbf{y} ,
- a (possibly empty) subset J of \mathcal{S} .

The corresponding algebra element is $\phi^{\mathbf{x}, \mathbf{y}}(p) \cdot \prod_{j \in J} C_j$. We call these basis vectors *pure algebra elements*.

3.4. Gradings. Note first since $\mathcal{B}(m, k)$ is obtained as a quotient of $\mathcal{B}_0(m, k)$ by a w -homogeneous ideal, the w -grading by $(\frac{1}{2}\mathbb{Z})^m$ descends to a grading on $\mathcal{B}(m, k)$. The w -gradings extend to $\mathcal{B}(m, k, \mathcal{S})$, by declaring $w_i(C_j) = 1$ if $i = j$ and 0 otherwise.

Explicitly, for $1 \leq i \leq m$, the i^{th} weight w_i of L_i and R_i is $1/2$, and w_i of U_i and C_i is 1, and w_i vanishes on L_j , R_j , U_j , and C_j with $j \neq i$. These functions each induce gradings on $\mathcal{B}(m, k, \mathcal{S})$; i.e. if a and b are homogenous elements with $a \cdot b \neq 0$, then $w_i(a \cdot b) = w_i(a) + w_i(b)$.

We will be especially interested in the following specialization, called the *Alexander grading*:

$$(3.6) \quad \text{Alex}(a) = - \sum_{s \in \mathcal{S}} w_s(a) + \sum_{t \notin \mathcal{S}} w_t(a).$$

For $i = 1, \dots, m$, there is a filtration \mathbf{m}_i $\mathcal{B}(m, k, \mathcal{S})$ with values in $\{0, 1\}$, specified by the function on pure algebra elements b so that $\mathbf{m}_i(b) = 1$ if b is divisible by C_i and 0 otherwise. This extends to a filtration on the algebra:

$$\mathcal{B}(m, k, \mathcal{S}) = \mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=0} \oplus \mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=1},$$

where $\mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=c}$ for $c = 0$ or 1 is the vector space spanned by generators $a \in \mathcal{B}(m, k, \mathcal{S})$ with $\mathbf{m}_i(a) = c$. We call this a filtration, because the differential on the algebra d preserves $\mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=0}$, but not $\mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=1}$. Elements of $\mathcal{B}(m, k, \mathcal{S})$ that are contained entirely in $\mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=0}$ or $\mathcal{B}(m, k, \mathcal{S})_{\mathbf{m}_i=1}$ are called \mathbf{m}_i -homogeneous. Note that $\sum_{i=1}^m \mathbf{m}_i$ induces a \mathbb{Z} -grading on $\mathcal{B}(m, k, \mathcal{S})$ that drops

by one under the differential. We will be interested in a different normalization of this, the *Maslov grading*, defined by

$$(3.7) \quad \mathbf{m}(a) = \#(C_i \text{ in } a) - 2 \sum_{s \in \mathcal{S}} w_s(a) = \sum_{i=1}^m \mathbf{m}_i(a) - 2 \sum_{s \in \mathcal{S}} w_s(a).$$

3.5. Examples. When $k = 0$, $\mathcal{B}(m, 0) \cong \mathbb{F}$; where $1 = \mathbf{I}_\emptyset$. The elements L_i , R_i , and U_i are zero. When $k = m + 1$

$$\mathcal{B}(m, m + 1) = \mathbb{F}[U_1, \dots, U_m];$$

there is only one idempotent $1 = \mathbf{I}_{\{0, \dots, m\}}$.

Some other examples can be illustrated by the use of path algebras. Given a directed graph Γ , the *path algebra* is the \mathbb{F} -vector space generated by sequences of edges $e_1 * \dots * e_n$ where the terminal vertex of e_i is the initial vertex of e_{i+1} . We include also trivial paths, based at any vertex. If two paths can be concatenated, then their product is the concatenation; otherwise it is zero.

For example, consider Figure 7, which is a graph with two vertices and four edges. Two of the edges (closed loops) are labelled by U , but they are not the same, as they have different initial points. Identifying $L * R$ and $R * L$ with the corresponding closed loops labelled by U , we obtain the algebra $\mathcal{B}(1, 1)$.

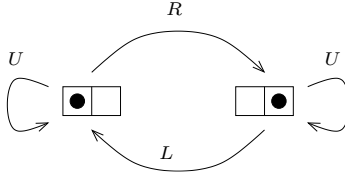


FIGURE 7. **Picture of $\mathcal{B}(1, 1)$.** The two idempotents $\mathbf{I}_{\{0\}}$ and $\mathbf{I}_{\{1\}}$ are pictured, with arrows corresponding to algebra elements connecting idempotents. The algebra $\mathcal{B}(1, 1)$ is the quotient of the pictured path algebra, where $R * L$ and $L * R$ are identified with the two closed loops labelled by U (that are distinguished by their starting points).

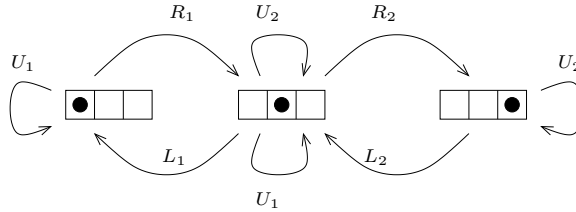


FIGURE 8. **Picture of $\mathcal{B}(2, 1)$.** The three idempotents $\mathbf{I}_{\{0\}}$, $\mathbf{I}_{\{1\}}$, and $\mathbf{I}_{\{2\}}$ are pictured, with arrows corresponding to algebra elements connecting idempotents. The algebra $\mathcal{B}(2, 1)$ can be thought of as a quotient of the pictured path algebra, divided out by relations $R_1 * R_2 = 0$, $L_2 * L_1 = 0$, $R_i * L_i = U_i$, $L_i * R_i = U_i$, $U_1 * U_2 = U_2 * U_1$.

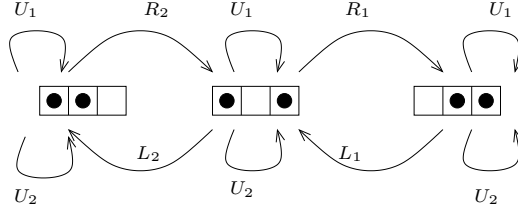


FIGURE 9. **Picture of $\mathcal{B}(2,2)$.** The three idempotents $\mathbf{I}_{\{0,1\}}$, $\mathbf{I}_{\{0,2\}}$, and $\mathbf{I}_{\{1,2\}}$ are pictured, with arrows corresponding to algebra generators. The algebra $\mathcal{B}(2,2)$ can be thought of as a quotient of the pictured path algebra by $R_i * L_i = U_i$, $L_i * R_i = U_i$, $U_1 * L_2 = L_2 * U_1$, $U_1 * R_2 = R_2 * U_1$, $U_2 * L_1 = L_1 * U_2$, and $U_2 * R_1 = R_1 * U_2$.

3.6. Symmetries in the algebras. Consider the map $\rho: \{0, \dots, m\} \rightarrow \{0, \dots, m\}$ with $\rho(i) = m - i$. There is a map

$$(3.8) \quad \mathcal{R}: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \rho'_m(\mathcal{S}))$$

where $\rho'_m(i) = m + 1 - i$, characterized by the following properties. First,

$$\mathcal{R}(\mathbf{I}_{\mathbf{x}}) = \mathbf{I}_{\rho(\mathbf{x})};$$

and if $a \in \mathcal{B}(m, k)$ is non-zero and homogeneous with specified weights

$$(w_1(a), \dots, w_m(a)),$$

then $\mathcal{R}(a) = b$ is the non-zero element that is homogeneous with specified weights $w_i(b) = w_{m+1-i}(a)$. We extend this to $\mathcal{B}(m, k, \mathcal{S}) \supset \mathcal{B}(m, k)$ by requiring

$$\mathcal{R}(C_j \cdot a) = C_{m+1-j} \cdot \mathcal{R}(a).$$

Note that for all $i = 1, \dots, m$ and $j \in \mathcal{S}$,

$$\mathcal{R}(L_i) = R_{m+1-i} \quad \mathcal{R}(R_i) = L_{m+1-i} \quad \mathcal{R}(U_i) = U_{m+1-i} \quad \mathcal{R}(C_j) = C_{m+1-j}$$

Clearly, this induced map \mathcal{R} induces an isomorphism of algebras.

Another symmetry identifies the algebra with its “opposite” algebra. Specifically, the map $o(\mathbf{I}_{\mathbf{x}}) = \mathbf{I}_{\mathbf{x}}$ extends to an isomorphism of rings

$$(3.9) \quad o: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \mathcal{S})^{\text{op}},$$

with

$$o(L_i) = R_i \quad o(R_i) = L_i \quad o(U_i) = U_i \quad o(C_j) = C_j.$$

3.7. Canonical DD -bimodules. Let

$$(3.10) \quad \mathcal{B}_1 = \mathcal{B}(m, k_1, \mathcal{S}_1), \quad \mathcal{B}_2 = \mathcal{B}(m, k_2, \mathcal{S}_2)$$

where $k_1 + k_2 = m + 1$ and $\mathcal{S}_2 = \{1, \dots, m\} - \mathcal{S}_1$.

Note that there is a natural one-to-one correspondence between the I -states for \mathcal{B}_1 and \mathcal{B}_2 : if $\mathbf{x} \subset \{0, \dots, m\}$ is a k_1 -element subset, then its complement \mathbf{x}' is a k_2 -element subset of $\{0, \dots, m\}$. In this case, we say that \mathbf{x} and \mathbf{x}' are *complementary I -states*.

A DD bimodule over $\mathcal{B}_1 - \mathcal{B}_2$ is specified as follows. Let \mathcal{K} be the \mathbb{F} -vector space whose generators $k_{\mathbf{x}}$ correspond to I -states for $\mathcal{B}(m, k_1, \mathcal{S}_1)$. We give \mathcal{K} the

structure of a left module over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_2)$, so that the action of $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_2)$ is specified by

$$(\mathbf{I}_{\mathbf{y}} \otimes \mathbf{I}_{\mathbf{w}}) \cdot k_{\mathbf{x}} = \begin{cases} k_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{y} \text{ and } \mathbf{w} \text{ is complementary to } \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

The algebra element

$$A = \sum_{i=1}^m (L_i \otimes R_i + R_i \otimes L_i) + \sum_{s \in \mathcal{S}_1} C_s \otimes U_s + \sum_{t \in \mathcal{S}_2} U_t \otimes C_t \in \mathcal{B}_1 \otimes \mathcal{B}_2$$

specifies a map

$$\delta^1: \mathcal{K} \rightarrow \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{K}.$$

by $\delta^1(v) = A \otimes v$ (where the tensor product is taken over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_2)$).

This data can be represented graphically, as follows. Take m vertical segments, and orient them arbitrarily. The upwards pointing segments specify \mathcal{S}_2 and the downwards pointing ones specify \mathcal{S}_1 . We draw a horizontal arc crossing each of the vertical segment as a placeholder, and keep only half of each vertical segment above (resp. below) the horizontal arc if the segment is oriented upwards (resp. downwards). The horizontal arc is divided into $m+1$ intervals by the vertical arcs. An element of $k_{\mathbf{x}}$ is represented as a collection of $m+1$ dark dots corresponding to the intervals in the horizontal segment, each of distributed either above or below the horizontal segment. The set of intervals whose dots are below give \mathbf{x} . When illustrating generators of $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{K}$, we draw the algebra element from \mathcal{B}_1 below the diagram for $k_{\mathbf{x}}$ and the algebra element from \mathcal{B}_2 above. See Figure 10 for an illustration.

Lemma 3.9. *The map δ^1 satisfies the type DD structure relation.*

Proof. This is equivalent to the statement that

$$dA + A \cdot A = 0,$$

thought of as an element of $\mathcal{B}_1 \otimes \mathcal{B}_2$.

We consider the four types of terms A :

- $L_i \otimes R_i$
- $R_i \otimes L_i$
- $C_i \otimes U_i$ when $i \in \mathcal{S}_1$
- $U_i \otimes C_i$ when $i \in \mathcal{S}_2$.

In $dA + A \cdot A$, some of the terms cancel since U_i and C_i are in the center; other terms cancel when the indices i and j are sufficiently far ($|i-j| > 1$). When $|i-j| = 1$, terms of the first type and the second type cancel. When $i = j$, a term of the first and second type cancel with differentials of terms of the third or fourth type. \square

The two out-going algebras \mathcal{B}_1 and \mathcal{B}_2 are graded by $\frac{1}{2}\mathbb{Z}^m$,

$$\mathbf{gr}_{\mathcal{B}_1}(a_1) = (w_1(a_1), \dots, w_m(a_1)) \quad \mathbf{gr}_{\mathcal{B}_2}(a_2) = (w_1(a_2), \dots, w_m(a_2));$$

$$\begin{aligned}
\delta^1 \begin{array}{c} \bullet \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} U_2 \\ \bullet \bullet \\ | \\ C_2 \bullet \end{array} \\
\delta^1 \begin{array}{c} \bullet \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} C_1 \\ \bullet \bullet \\ | \\ U_1 \end{array} + \begin{array}{c} U_2 \\ \bullet \bullet \\ | \\ C_2 \end{array} \\
\delta^1 \begin{array}{c} \bullet \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} C_1 \\ \bullet \bullet \\ | \\ U_1 \end{array} + \begin{array}{c} U_2 \\ \bullet \bullet \\ | \\ C_2 \end{array}
\end{aligned}$$

FIGURE 10. **The canonical DD bimodule.** In the left column, we have the three generators for the DD bimodule, with $\mathcal{B}_1 = \mathcal{B}(2, 1, \{2\})$ and $\mathcal{B}_2 = \mathcal{B}(2, 2, \{1\})$. They correspond to the I -states $\{2\}$, $\{1\}$, and $\{0\}$ respectively. To the right we have the non-zero terms in the differential. To read off the algebra element in \mathcal{B}_1 , reflect the picture vertically. For example, if the three generators on the left are denoted E , F , and G , then the first equation expresses $\delta^1(E) = (L_2 \otimes R_2) \otimes F + (C_2 \otimes U_2) \otimes E$.

the canonical DD bimodule is also graded by $\frac{1}{2}\mathbb{Z}^m$ where $(v_1, v_2) \in \frac{1}{2}\mathbb{Z}^m \oplus \frac{1}{2}\mathbb{Z}^m$ acts on $w \in \frac{1}{2}\mathbb{Z}^m$ by $(v_1, v_2) \cdot w = (v_1 - v_2 + w)$. In fact, the module is supported in grading 0; i.e. for each term $a_1 \otimes a_2$ in A specifying δ^1 , $\mathbf{gr}_{\mathcal{B}_2}(a_2) = \mathbf{gr}_{\mathcal{B}_1}(a_1)$.

3.8. The canonical DD bimodule is invertible.

3.8.1. *A candidate for the inverse module.* The main result is to prove that the bimodule defined in the previous section is invertible.

We consider

$$Y_{\mathcal{B}_1, \mathcal{B}_2} = \text{Mor}^{\mathcal{B}_1}((\mathcal{B}_2(\mathcal{B}_2)_{\mathcal{B}_2}) \boxtimes_{\mathcal{B}_2} (\mathcal{B}_1, \mathcal{B}_2 \mathcal{K}), {}^{\mathcal{B}_1} \mathbb{I}_{\mathcal{B}_1}).$$

This is naturally a $\mathcal{B}_2 - \mathcal{B}_1$ -bimodule (of type AA , with both actions on the right).

As a vector space, Y is spanned by elements of the form $(\bar{a}|b)$, where $\bar{a} \in \overline{\mathcal{B}_2}$ and $b \in \mathcal{B}_1$, where here $\overline{\mathcal{B}}$ is opposite bimodule to \mathcal{B} , thought of as a bimodule (as in Equation (2.6)), subject the restriction that the left idempotent of \bar{a} is complementary to the left idempotent of b . Recall that $\overline{\mathcal{B}_2}$ is the \mathcal{B}_2 -bimodule consisting of maps from \mathcal{B}_2 to \mathbb{F} . This is a left-right $\mathcal{B}_2 - \mathcal{B}_2$ bimodule by the rule

$$x \cdot \bar{a} \cdot y = (\xi \mapsto \bar{a}(y \cdot \xi \cdot x)),$$

for $x, y \in \mathcal{B}_2$ and $\bar{a} \in \overline{\mathcal{B}_2}$.

We can take these generating vectors so that \bar{a} is dual to a generating algebra element in \mathcal{B}_2 ; note that the right idempotent of a is the left idempotent of its dual \bar{a} .

The differential on Y has terms

$$(\overline{a}|b) \mapsto (L_i \cdot \overline{a}|R_i \cdot b)$$

$$(\overline{a}|b) \mapsto (R_i \cdot \overline{a}|L_i \cdot b)$$

and furthermore

$$(\overline{a}|b) \mapsto (U_i \cdot \overline{a}|C_i \cdot b)$$

if $i \in \mathcal{S}_1$; otherwise

$$(\overline{a}|b) \mapsto (C_i \cdot \overline{a}|U_i \cdot b).$$

Finally,

$$(\overline{a}|C_i \cdot b) \mapsto (\overline{a}|U_i \cdot b)$$

and

$$(\overline{U_i \cdot a}|b) \mapsto (\overline{C_i \cdot a}|b).$$

Note that the action by $\mathcal{B}_2 - \mathcal{B}_1$ is given by

$$(\overline{a}|b) \cdot (b_2 \otimes b_1) = (\xi \mapsto \overline{a}(b_2 \cdot \xi)|b \cdot b_1)$$

We draw pictures of this action as follows. We draw a pair $(\overline{a}|b)$ where a and b are pure algebra elements, by drawing first a graphical representation of a (dual to \overline{a}) on top of a graphical representation for b . In this picture, the right idempotent of b is on the bottom, and the right idempotent of \overline{a} is the initial state (on the top) of a . See Figure 11.

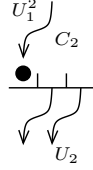


FIGURE 11. **Graphical representative of elements of Y .**

Here we have a picture of $(\overline{L_1 \cdot U_1^2 \cdot C_2}|L_1 \cdot L_2 \cdot U_2)$ in Y , thought of as an element of $\overline{\mathcal{B}_2(2, 1, \{1, 2\})} \otimes \mathcal{B}_1(2, 2, \emptyset)$.

3.8.2. *An example:* Consider $\mathcal{B}_1 = \mathcal{B}(1, 1, \{1\})$ and $\mathcal{B}_2 = \mathcal{B}(1, 1, \emptyset)$. The algebra \mathcal{B}_2 has generators L , R , and U , and idempotents $\mathbf{I}_{\{0\}}$ and $\mathbf{I}_{\{1\}}$. Moreover,

$$\mathbf{I}_{\{0\}} \cdot R \cdot \mathbf{I}_{\{1\}} = R \quad \mathbf{I}_{\{1\}} \cdot L \cdot \mathbf{I}_{\{0\}} = L,$$

while \mathcal{B}_1 has the same generators, and an additional C . The bimodule decomposes into four summands, according to the right idempotent of $(\overline{a}|b)$.

In right idempotent $\mathbf{I}_{\{1\}} \otimes \mathbf{I}_{\{0\}}$, the complex further decomposes into summands. One of these summands contains the single element $(\overline{\mathbf{I}_{\{1\}}}| \mathbf{I}_{\{0\}})$. Another summand is the square

$$(3.11) \quad \begin{array}{ccccc} & & (\overline{L}|L) & & \\ & \nearrow & & \searrow & \\ (\overline{\mathbf{I}_{\{1\}} \cdot U}| \mathbf{I}_{\{0\}}) & & & & (\overline{\mathbf{I}_{\{1\}}}| U \cdot \mathbf{I}_{\{0\}}) \\ & \searrow & & \nearrow & \\ & & (\overline{\mathbf{I}_{\{1\}}}| C \cdot \mathbf{I}_{\{0\}}) & & \end{array}$$

See Figure 12 for a picture. (In fact, there are infinitely many different summands.)

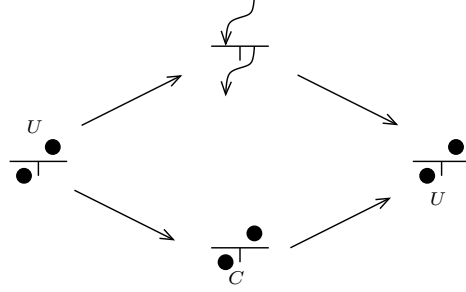


FIGURE 12. Terms in the differential of Y . We have drawn pairs of algebra elements; the top algebra element should be dualized to get the corresponding generator.

In right idempotent $\mathbf{I}_{\{0\}} \otimes \mathbf{I}_{\{0\}}$, there is a collection of acyclic complexes. For example, in the portion with total weight $1/2$, we have:

$$(3.12) \quad (\overline{R}|\mathbf{I}_{\{0\}}) \longrightarrow (\overline{\mathbf{I}_{\{0\}}}|L).$$

In the portion with total weight $3/2$, we have:

$$(3.13) \quad \begin{array}{ccccccc} & & & & (\overline{R}|C \cdot \mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}}|L \cdot C) \\ & & \nearrow & & \downarrow & \nearrow & \downarrow \\ (\overline{R}|\overline{U}|\mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}}|\overline{U}|L) & \longrightarrow & (\overline{R}|U \cdot \mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}}|L \cdot U). \end{array}$$

3.8.3. The candidate is the inverse. We show now that the candidate inverse from Section 3.8.1 is indeed the inverse for the canonical type DD bimodule, by verifying the hypotheses of Lemma 2.16. Continuing notation from earlier, the algebras \mathcal{B}_1 and \mathcal{B}_2 are as in Equation (3.10); Y is as in Section 3.8.1, generated by pairs $(\overline{a}|b)$, with $a \in \mathcal{B}_2$ and $b \in \mathcal{B}_1$.

Proposition 3.10. *The rank of the homology group of Y is $\binom{m+1}{k}$; it is generated by the elements of the form $(\overline{\mathbf{I}_{\mathbf{x}'}}|\mathbf{I}_{\mathbf{x}})$, where \mathbf{x} and \mathbf{x}' are complementary idempotents.*

Lemma 3.11. *The candidate complex Y decomposes into a direct sum of complexes $C(Z, \mathbf{x}, \mathbf{y})$, indexed by idempotent states \mathbf{x} and \mathbf{y} and $Z \in (\frac{1}{2}\mathbb{Z})^m$, where $C(Z, \mathbf{x}, \mathbf{y})$ is the vector space generated by pairs $(\overline{a}|b)$ where a and b are pure algebra elements with $\mathbf{I}_{\mathbf{x}} \cdot a = a$ and $b \cdot \mathbf{I}_{\mathbf{y}} = b$, and $w(a) + w(b) = Z$.*

Proof. Since the left idempotent of a is the right idempotent of \overline{a} , the splitting by idempotents corresponds to the splitting of Y according to right idempotents. The fact that the splitting by Z is well defined follows immediately from the definition of ∂ . (More abstractly, it is a formal consequence of the fact that δ^1 on the

tautological DD bimodule is specified by an algebra element $A = \sum a_i \otimes b_i$ where the weight of a_i equals the weight of b_i . \square

We can write the differential ∂ on Y as a sum of terms $\partial = \sum_{i=1}^m \partial_i$, where ∂_i involves terms on the i^{th} strand. More precisely, there is a differential $d_i: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \mathcal{S})$ that vanishes unless $i \in \mathcal{S}$, in which case $d_i(C_i) = U_i$, but $d_i(C_j) = 0$ for all $j \neq i$ and $d_i(L_j) = d_i(U_j) = d_i(R_j) = 0$ for all $j = 1, \dots, m$. Define

$$\partial_i(\bar{a}|b) = \begin{cases} (\bar{a}|d_i b) + (R_i \cdot \bar{a}|L_i \cdot b) + (L_i \cdot \bar{a}|R_i \cdot b) + (U_i \cdot \bar{a}|C_i \cdot b) & \text{if } i \in \mathcal{S}_1 \\ (\bar{d}_i \bar{a}|b) + (R_i \cdot \bar{a}|L_i \cdot b) + (L_i \cdot \bar{a}|R_i \cdot b) + (C_i \cdot \bar{a}|U_i \cdot b) & \text{if } i \in \mathcal{S}_2, \end{cases} \quad (3.14)$$

where $\bar{d}_i: \bar{\mathcal{B}}_2 \rightarrow \bar{\mathcal{B}}_2$ is dual to the differential $d_i: \mathcal{B}_2 \rightarrow \mathcal{B}_2$.

Lemma 3.12. *The differential ∂ on Y can be written as $\partial = \sum_{i=1}^m \partial_i$, where $\partial_i^2 = 0$ and $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for $i \neq j$.*

Proof. This can be seen directly. A little more conceptually, on homogeneous generators $(\bar{a}|b)$ where a and b are pure algebra elements, the functions $w_j(a)$ for $j = 1, \dots, m$ and

$$\mathbf{m}_j(\bar{a}|b) = \begin{cases} \mathbf{m}_j(b) & \text{if } j \in \mathcal{S}_1 \\ 1 - \mathbf{m}_j(a) & \text{if } j \in \mathcal{S}_2, \end{cases} \quad (3.15)$$

for $j = 1, \dots, m$ induce a filtration on $C(Z, \mathbf{x}, \mathbf{y})$ by $(\frac{1}{2}\mathbb{Z})^m \times \{0, 1\}^m$. Restricting to values with $j \neq i$, we have a filtration on $C(Z, \mathbf{x}, \mathbf{y})$ with $(\frac{1}{2}\mathbb{Z})^{m-1} \times \{0, 1\}^{m-1}$. The associated graded object is clearly $(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$. \square

Proposition 3.13. *Fix idempotents \mathbf{x} and \mathbf{y} and a total weight Z ; suppose moreover that the weight Z is non-zero. Then there is a position i with the property that $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_i) = 0$*

Proposition 3.13 is proved using the following lemma:

Lemma 3.14. *The chain complex $(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$ splits into subcomplexes indexed by $(\frac{1}{2}\mathbb{Z})^{m-1} \times \{0, 1\}^{m-1}$ that are spanned by $(\bar{a}|b)$, where a and b are pure algebra elements whose weights $w_j(b)$ with $j \neq i$ and values $\mathbf{m}_j(\bar{a}|b)$ (as defined in Equation (3.15) above) for all $j \neq i$ are specified.*

Proof. This follows from the form of ∂_i : it does not involve any of the C_j with $j \neq i$, and it does not change the weight of b away from i . \square

Lemma 3.15. *For fixed Z and $i \in \{1, \dots, m\}$, \mathbf{x} , \mathbf{y} and $(\frac{1}{2}\mathbb{Z})^{m-1} \times \{0, 1\}^{m-1}$, consider the corresponding subcomplex C of $(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$ as in Lemma 3.14. Then $H(C) \neq 0$ precisely when there is a single element in C . When the homology of C is non-zero, there are pure algebra elements a and b , and a single generator $(\bar{a}|b)$ in C ; and one of the following holds:*

- $w_i(a) = w_i(b) = 0$
- $w_i(a) + w_i(b) = \frac{1}{2}$
- $w_i(a) + w_i(b) = 1$ and either $a = a_0 \cdot C_i$ or $b = b_0 \cdot C_i$.

In particular, writing $z_i = w_i(a) + w_i(b)$, if $H(C(Z, \mathbf{x}, \mathbf{y}), \partial_i) \neq 0$, then $z_i \leq 1$.

Proof. When $z_i = 0$, the statement is clear; otherwise, there are cases, according to the local picture of the summand of $C(Z, \mathbf{x}, \mathbf{y})$ near i . Specifically, if $C(Z, \mathbf{x}, \mathbf{y})$ contains an element $(\bar{a}|b)$ with $a = \mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{w}}$, we consider separately the following two cases:

- (C-1) $i - 1, i \in \mathbf{w}$.
- (C-2) Exactly one of $i - 1$ or i is in \mathbf{w} .

There is a third case, where neither $i - 1, i \in \mathbf{w}$; but that is symmetric to Case (C-1) by a symmetry that exchanges roles of \mathcal{B}_1 and \mathcal{B}_2 , and then dualizes C .

There are five further subcases of Case (C-1), as follows. if $i - 1$ and i are the j^{th} and $j + 1^{st}$ terms in the sequence \mathbf{w} , we further subdivide according to the placement of the j^{th} and the $j + 1^{st}$ terms in the sequence of \mathbf{x} . Since \mathbf{x} and \mathbf{w} are not too far, the pair $\{j, j + 1\}$ must be one of $\{i - 2, i - 1\}$, $\{i, i + 1\}$, $\{i - 2, i\}$, $\{i - 1, i + 1\}$, $\{i - 1, i\}$, $\{i - 2, i + 1\}$.

The cases where $\{j, j + 1\} = \{i - 2, i\}$ and $\{i - 1, i + 1\}$ are exchanged by reflection of through a vertical axis (compare the map \mathcal{R} from Equation (3.8)), as are $\{i - 2, i - 1\}$ and $\{i, i + 1\}$. Dropping two symmetric cases, we arrive at the four pictures in the first three columns of Figure 13.

Similarly, in Case (C-2), if $i - 1 \in \mathbf{w}$ (which can be arranged after a vertical reflection) is the j^{th} term in \mathbf{w} and i is the k^{th} one in the left idempotent of b , then we further subdivide according to the placement of the j^{th} and k^{th} terms in \mathbf{x} and \mathbf{y} respectively. These possibilities (after eliminating symmetric duplicates) are represented in the remaining pictures in Figure 13.

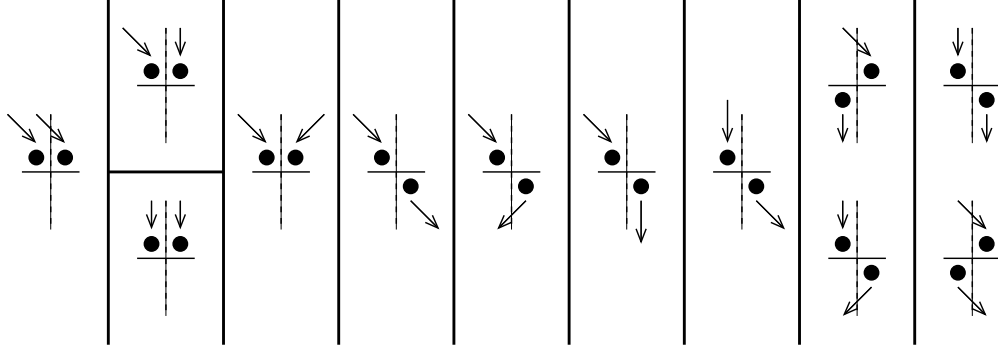


FIGURE 13. $C(Z, \mathbf{x}, \mathbf{y})$ cases.

We consider now Cases (C-1) and (C-2) separately, and further subdivide them into subcases (that are related to the cases in Figure 13), verifying the lemma in each subcase.

Case (C-1), with $z_i = 1/2$. (The first column in Figure 13.) There is now a single generator of this type, of the form $(\bar{a}|b)$ with $w_i(a) = 1/2$ and $w_i(b) = 0$. so the homology is one-dimensional.

Case (C-1), with $z_i \notin \mathbb{Z}$ and $z_i > 1/2$. (Again, this is illustrated in first column in Figure 13.) This has two further subcases, according to whether $i \in \mathcal{S}_1$ or \mathcal{S}_2

(i.e. whether $C_i \in \mathcal{B}_1$ or \mathcal{B}_2). If $i \in \mathcal{S}_2$, there are two generators of the complex, $(\overline{aU_i^k}|b)$ and $(\overline{aU_i^{k-1}C_i}|b)$ where $k = z_i - \frac{1}{2} \geq 1$, and a and b are fixed pure algebra elements with $w_i(a) = \frac{1}{2}$, $w_i(b) = 0$. To see that both types of terms appear, note that under the present hypotheses on $a = a \cdot \mathbf{I}_w$ and w , if $a \neq 0$ then $U_i \cdot a \neq 0$, as well. Clearly,

$$\partial_i(\overline{aU_i^k}|b) = (\overline{aU_i^{k-1}C_i}|b),$$

so the homology of the corresponding complex vanishes. When $i \notin \mathcal{S}_2$, the terms are of the form $(\overline{aU_i^n}|b)$ and $(\overline{aU_i^{n-1}C_i}|b)$, and the homology again vanishes.

Case (C-1), with $z_i \in \mathbb{Z}$. When $i \notin \mathcal{S}_2$, we either have a single element $(\overline{a}|C_i b)$ with $w_i(a) = w_i(b) = 0$ (as in the third column of Figure 13; this can also appear in the second column), or elements of the form $(\overline{aU_i^j}|b)$ and $(\overline{aU_i^j}|C_i b)$, with fixed a, b so that $w_i(a) = w_i(b) = 0$, where $j = z_i$ and the differential is given by

$$\partial_i(\overline{aU_i^j}|b) = (\overline{aU_i^{j-1}C_i}|b);$$

so the homology is trivial. The case where $i \in \mathcal{S}_2$ works similarly. We either have a single element $(\overline{aC_i}|b)$, $w_i(a) = w_i(b) = 0$ or elements of the form $(\overline{aU_i^j}|b)$ and $(\overline{aU_i^{j-1}C_i}|b)$. Once again, the differentials cancel out the homology.

Case (C-2), with $z_i = \frac{1}{2}$. (This is the fifth or the eighth column of Figure 13.) There are two possibly non-zero elements in the complex. These elements have the form $(\overline{a \cdot R_i}|b)$ and $(\overline{a}|L_i \cdot b)$, where $w_i(a) = w_i(b) = 0$. There are two subcases: in one subcase, only one of $a \cdot R_i$ or $L_i \cdot b$ is zero, so we have a single generator. (This case occurs in the fifth column of Figure 13.) In the other case, when both $a \cdot R_i$ and $L_i \cdot b$ are non-zero (which occurs now only in the eighth column of Figure 13), we have that

$$\partial_i(\overline{a \cdot R_i}|b) = (\overline{a}|L_i \cdot b),$$

so the homology is trivial.

Case (C-2), with $z_i \notin \mathbb{Z}$, and $z_i = \frac{1}{2} + t > \frac{1}{2}$. (These cases are illustrated in the fifth and eighth columns of Figure 13.) Suppose that $i \notin \mathcal{S}_2$. We can assume that all the generators a corresponding summand of $C(Z, \mathbf{x}, \mathbf{y})$ have the following form $(\overline{aR_iU_i^j}|bU_i^k)$, $(\overline{aR_iU_i^j}|bC_iU_i^{k-1})$, $(\overline{aU_i^j}|L_i bU_i^k)$, and $(\overline{aU_i^j}|L_i bC_iU_i^{k-1})$, for fixed a and b with $w_i(a) = w_i(b) = 0$, and $j + k + \frac{1}{2} = z_i$.

Note that at least one of $aR_i \neq 0$ or $L_i b \neq 0$, otherwise the chain complex would be trivial. If $aR_i \neq 0$ and $L_i b = 0$, then there are exactly two elements in the chain complex, $(\overline{aR_iU_i^t}|b)$ and $(\overline{aU_i^{t-1}C_i}|b)$, and the differential cancels them. (Again, cases of this kind can occur in either the fifth or the eighth columns of Figure 13.) If $aR_i = 0$ and $L_i b \neq 0$, there are once again two terms, $(\overline{a}, L_i bU_i^t)$ and $(\overline{a}, L_i bC_iU_i^{t-1})$, and these two terms cancel in the differential.

In the remaining case where $aR_i \neq 0$ and $L_i b \neq 0$, we show that $H(C(Z, \mathbf{x}, \mathbf{y}), \partial_i) = 0$, using a further filtration on the complex, induced by the function on pure algebra elements a and b that associates to $(\overline{a}|b)$ the i^{th} weight of b , $w_i(b)$. This induces a grading on the vector space underlying $C(Z, \mathbf{x}, \mathbf{y}) = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{F}_k$. Writing $\partial_0^i(\overline{a}|b) = (\overline{a}|d_i b)$, we have that

$$\partial_i = \partial_0^i + L,$$

where $\partial_0^i: \mathcal{F}_k \rightarrow \mathcal{F}_k$, and $L: \mathcal{F}_k \rightarrow \bigoplus_{\ell > k} \mathcal{F}_\ell$. (We are using the form of ∂_i from Equation (3.14).) Thus, the associated graded complex on $(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$ is equipped with the differential ∂_0^i , and we will show that its homology vanishes.

When $k \in \mathbb{Z}$, the complex \mathcal{F}_k is spanned by $(\overline{aR_iU_i^j}, bU_i^k)$ and $(\overline{aR_iU_i^j}, bC_iU_i^{k-1})$, and they are connected by a differential in ∂_0^i . Similarly, when $k \geq 1$, the complex $\mathcal{F}_{k+\frac{1}{2}}$ contains the two elements when $(\overline{aU_i^j}, L_i bU_i^k)$, and $(\overline{aU_i^j}, L_i bC_iU_i^{k-1})$, and these elements are connected by a differential in ∂_0^i . Thus, $H(\mathcal{F}_k, \partial_0^i) = 0$ except when $k = 0$ and $1/2$; and \mathcal{F}_0 is generated by the single element $(\overline{aR_iU_i^t}|b)$ and $\mathcal{F}_{1/2}$ is generated by the element $(\overline{aU_i^t}|L_i b)$. Thus, by a simple filtration argument, $H(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$ is computed as the homology of a two-dimensional vector space generated by these latter two generators $(\overline{aR_iU_i^t}|b)$ and $(\overline{aU_i^t}|L_i b)$; and these two elements are connected by a differential. This completes the verification that $H(C(Z, \mathbf{x}, \mathbf{y}), \partial_i) = 0$ in this case. See Equation (3.13) for an illustration when $z = 3/2$ (and observe that for the diagram, the horizontal coordinate in the plane measures the filtration considered here). The case where $i \in \mathcal{S}_2$ works similarly: the homology of ∂_0^i is now supported in \mathcal{F}_t and $\mathcal{F}_{t+1/2}$, with generators $(\overline{aR_i}|U_i^t b)$ and $(\overline{a}|L_i U_i^t b)$, with a cancelling differential.

Case (C-2), with $z_i = 1$. Assume that $i \notin \mathcal{S}_2$. The chain complex $C(Z, \mathbf{x}, \mathbf{y})$ is spanned by four vectors elements of the form

$$(\overline{a}|C_i b), \quad (\overline{a}|U_i b), \quad (\overline{aU_i}|b), \quad (\overline{aR_i}|R_i \cdot b),$$

where a and b are pure algebra elements with $w_i(a) = w_i(b) = 0$. One of the following must hold:

- All four elements are non-zero (which can occur in the ninth column of Figure 13). In this case,

$$\partial_i(\overline{aU_i}|b) = (\overline{aR_i}|R_i \cdot b) + (\overline{a}|C_i b)$$

$$\partial_i(\overline{a}|C_i b) = (\overline{a}|U_i b)$$

$$\partial_i(\overline{aR_i}|R_i \cdot b) = (\overline{a}|U_i b),$$

and the complex has trivial homology. (See Equation (3.11) for an example.)

- $a \cdot R_i = 0$ and $R_i \cdot b = 0$, in which case also $aU_i = bU_i = 0$, so three of the four vectors are zero, and the remaining vector $(\overline{a}|C_i b)$ generates the homology.
- $a \cdot R_i = 0$ but $R_i \cdot b \neq 0$. In this case $U_i \cdot b \neq 0$ (by Lemma 3.7), so two of the above elements $(\overline{a}|C_i b)$ and $(\overline{a}|U_i b)$ are non-zero. These two elements are connected by a differential, and the homology is trivial.
- $a \cdot R_i \neq 0$ and $R_i \cdot b = 0$. In this case, once again, two of the above elements are zero, and the remaining two are connected by a differential, so once again the homology is trivial.

The case where $i \in \mathcal{S}_2$ works similarly.

Case (C-2), with $z_i \in \mathbb{Z}$ and $z_i > 1$. Let $(\overline{a}|b)$ be some non-trivial generator in the complex, where $i \in \mathbf{w}$, so $i+1 \notin \mathbf{w}$. If either $a \cdot R_i = 0$ or $R_i \cdot b = 0$, then the argument works as in earlier cases (i.e. we have a pair of generators that cancel in

homology). Otherwise, as in the case where $z_i \notin \mathbb{Z}$ and $z_i > \frac{1}{2}$, we consider the filtration by the weight w_i of the b component. If $i \notin \mathcal{S}_2$, then for \mathcal{F}_k with $k \geq 1$, $H(\mathcal{F}_k, \partial_0^i) = 0$, since the complex \mathcal{F}_k has two terms in it that are connected by a differential in ∂_0^i . The remaining two terms are the generators of \mathcal{F}_k with $k = 0$ and $\frac{1}{2}$, which have the form $(\overline{aU_i^t}|b)$ and $(\overline{aU_i^{t-1}R_i}|R_i \cdot b)$; and these two terms are connected by a differential. The case where $i \in \mathcal{S}_2$ works similarly. \square

Proof of Proposition 3.13. Let $f: \{1, \dots, m+1\} \rightarrow \mathbb{Z}$ be defined by

$$f(i) = \#\{j | j < i \text{ and } j \in \mathbf{x}\} + \#\{j | j < i \text{ and } j \in \mathbf{y}\}$$

It is easy to see that $i - 1 \leq f(i) \leq i + 1$.

Choose n minimal so that the weight $z_n \neq 0$. There are three cases:

Case 1: $f(n) = n$. (Note that in this case, $z_i \in \mathbb{Z}$.) We claim that $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_n) = 0$. By Lemma 3.15, we need only consider cases where $z_n = 1$.

See Figure 14 when $n - 1 \in \mathbf{x}$; the cases where $n - 1 \in \mathbf{y}$ work similarly.

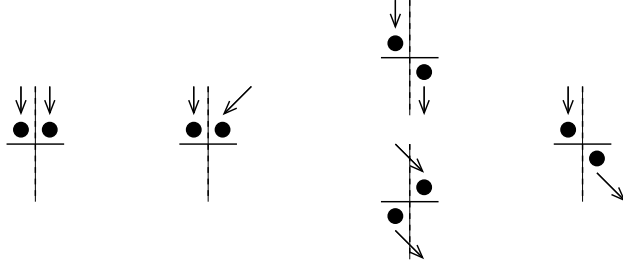


FIGURE 14. **Four cases where $f(n) = n$.** This is Case 1 from the proof of Proposition 3.13.

We claim that in all the cases from the figure, we will have an element of the form $(\overline{a \cdot U_n}|b)$ (where $w_n(a) = 0$) in the subcomplex. This is clear because in each case, either n is not contained in a generating interval, or the generating interval containing n also contains $n - 1$; but $w_{n-1}(a \cdot U_n) = 0$. Thus, Lemma 3.15 completes the case.

Case 2: $f(n) = n + 1$. (See Figure 15.) Again, we claim that $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_n) = 0$. Note that in this case, $z_n \in \frac{1}{2} + \mathbb{Z}$; so by Lemma 3.15, we can assume $z_n = \frac{1}{2}$. The two possible chain complex generators are $(\overline{a}|L_n \cdot b)$ and $(\overline{a \cdot R_n}|b)$. Since by assumption $z_i = 0$ for $i < n$, and clearly n is not the left endpoint of any generating interval (in either \mathcal{B}_1 or \mathcal{B}_2), it follows that both $aU_n \neq 0$ and $bU_n \neq 0$; and correspondingly $a \cdot R_n \neq 0$ and $L_n \cdot b \neq 0$; i.e. both generators are non-zero, so they cancel in homology.

Case 3: $f(n) = n - 1$. Since $f(m+1) = m+1$, we can find a minimal $j > n$ so that $f(j) = j$. Note that $z_{j-1} \equiv \frac{1}{2} \pmod{1}$. We distinguish four further subcases.

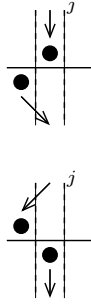
Case 3a: $z_{j-1} > \frac{1}{2}$. By Lemma 3.15, $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_{j-1}) = 0$.

Case 3b: $j < m+1$ and $z_{j-1} = \frac{1}{2}$ and $z_j = 0$. We claim that

$$H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_{j-1}) = 0.$$

FIGURE 15. $f(n) = n + 1$. This is Case 2 from the proof.

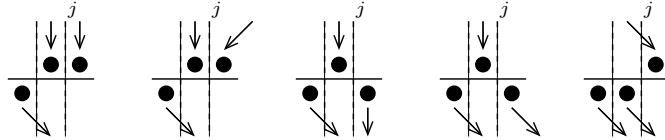
The two possible generators are of the form $(\overline{a}|R_j \cdot b)$ and $(\overline{a \cdot L_j}|b)$. Since $z_j = 0$,

FIGURE 16. $f(n) = n - 1$ and $z_{j-1} = \frac{1}{2}$.

it follows that $a \cdot U_{j-1} \neq 0$. and $b \cdot U_{j-1} \neq 0$, so both generators are non-zero, and cancel in homology.

Case 3c: $j = m + 1$ and $z_m = \frac{1}{2}$. Then $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_m) = 0$, exactly as in Case 3b.

Case 3d: $j < m + 1$ and $z_{j-1} = \frac{1}{2}$ and $z_j > 0$. We will show $H_*(C(Z, \mathbf{x}, \mathbf{y}), \partial_j) = 0$. We have illustrated five cases in Figure 17. The remaining cases are symmetric,

FIGURE 17. $f(n) = n - 1$, $z_{j-1} = \frac{1}{2}$, $z_j = 1$.

obtained by switching the roles of a and b .

In the first four of these five cases, observe that the corresponding chain complex contains a non-zero element of the form $(\overline{a \cdot U_j}|b)$ with $w_j(a) = w_j(b) = 0$. Thus, by Lemma 3.15, the homology is trivial (since the generating intervals on \mathcal{B}_2 containing j also contain $j - 1$). In the final case, the homology is also trivial by Lemma 3.15, since C_j is not present in the displayed generator. \square

Proof of Proposition 3.10. Decompose $H(Y)$ into the summands $C(Z, \mathbf{x}, \mathbf{y})$ as before. Suppose that the total weight Z is non-zero, and choose i as in Proposition 3.13. As in the proof of Lemma 3.12, there is a filtration on the complex $C(Z, \mathbf{x}, \mathbf{y})$ of Y , whose associated graded object is $(C(Z, \mathbf{x}, \mathbf{y}), \partial_i)$. It follows at once by an elementary spectral sequence argument that $H(C(Z, \mathbf{x}, \mathbf{y}), \partial) = 0$ if Z is a non-zero weight vector.

It remains to consider summands where the weight is zero. These correspond to pairs of complementary idempotents, equipped with a vanishing differential. There are, of course, $\binom{m+1}{k}$ such complementary pairs, as claimed. \square

Theorem 3.16. *The module Y is a quasi-inverse of the type DD bimodule \mathcal{K} .*

Proof. Note that \mathcal{B} is positively graded over \mathbf{k} , using the grading set $\Lambda = (\frac{1}{2}\mathbb{Z})^m$. Property (K-2) from Lemma 2.16 was verified in Proposition 3.10; Property (K-3) is obvious from the form of \mathcal{K} ; Lemma 2.16 now applies. \square

3.9. Grading sets associated to one-manifolds. Our knot invariant will be constructed by tensoring together bimodules with a particular kind of grading set. We formalize these grading sets presently, and study the boundedness needed for forming the tensor product.

Let W be an oriented disjoint union of finitely many intervals, equipped with a partition of its boundary $\partial W = Y_1 \cup Y_2$ into two sets of points. Let Y_i consist of m_i points. Let s_i denote the number of intervals in W that connect Y_i to itself, and s_0 denote the number of intervals that connect Y_1 to Y_2 in W . Let \mathcal{S}_1 be those points in Y_1 for which the oriented boundary of W appears with positive multiplicity in the oriented boundary of W , and let \mathcal{S}_2 be those points in Y_2 for which the oriented boundary of W appears with negative multiplicity in W . Choose any integer $0 \leq s \leq s_0 + 1$. Let $\mathcal{B}_1 = \mathcal{B}(m_1, s + s_1, \mathcal{S}_1)$, $\mathcal{B}_2(m_2, s + s_2, \mathcal{S}_2)$.

We can think of the Alexander multi-grading of \mathcal{B}_i as $H^0(Y_i; \mathbb{Q})$: the weights of algebra elements are functions on the points in Y_i . The sum of grading groups $H^0(Y_1; \mathbb{Q}) \oplus H^0(Y_2; \mathbb{Q}) = H^0(\partial W; \mathbb{Q})$ act on $H^1(W, \partial W; \mathbb{Q})$, via the coboundary map $d^0: H^0(\partial W) \rightarrow H^1(W, \partial W)$.

Remark 3.17. *In fact, the grading on the algebras is supported in $H^0(Y_i; \frac{1}{2}\mathbb{Z}) \subset H^0(Y_i; \mathbb{Q})$. Also, the grading set for our modules is contained in $H^1(W, \partial W; \frac{1}{4}\mathbb{Z})$; compare Equation (4.5).*

Definition 3.18. *Fix W as above. A type DA bimodule over $X = {}^{\mathcal{B}_2}X_{\mathcal{B}_1}$ is called adapted to W if it is graded by the grading set $H^1(W, \partial W)$ described above, and it is finite dimensional (as a vector space).*

Proposition 3.19. *Let W_1 be a disjoint union of finitely many intervals joining Y_1 to Y_2 ; and let W_2 be a disjoint union of finitely many intervals joining Y_2 to Y_3 . Suppose moreover that $W_1 \cup W_2$ has no closed components, i.e. it is a disjoint union of finitely many intervals joining Y_1 to Y_3 . Given any two bimodules ${}^{\mathcal{B}_2}X_{\mathcal{B}_1}^1$ and ${}^{\mathcal{B}_3}X_{\mathcal{B}_2}^2$ adapted to W_1 and W_2 respectively, we can form their tensor product ${}^{\mathcal{B}_3}X_{\mathcal{B}_2}^2 \boxtimes {}^{\mathcal{B}_2}X_{\mathcal{B}_1}^1$ (i.e. the infinite sums in its definition are finite); and moreover, it is a bimodule that is adapted to $W_1 \cup W_2$.*

Proof. Recall that the grading set of $X^2 \boxtimes X^1$ is $H^1(W_2, Y_3 \cup Y_2) \oplus H^1(W_1, Y_2 \cup Y_1)$ modulo the coboundary of $H^0(Y_2)$, which is identified with $H^1(W_2 \cup W_1, Y_3 \cup Y_1)$. Clearly, the tensor product is finite-dimensional.

Next, we argue the necessary finiteness. Fix algebra elements $(a_1, \dots, a_\ell) \in \mathcal{B}_1$. Fix generators $\mathbf{x}_1, \mathbf{y}_1$ for X^1 and $\mathbf{x}_2, \mathbf{y}_2$ for X^2 . Suppose that $(b_1 \otimes \dots \otimes b_j) \otimes \mathbf{y}_1$ appears in $\delta_i^j(\mathbf{x}_1, a_1, \dots, a_\ell)$. Since X^1 is graded, for each point i in Y_2 that is matched by W_1 to a point in Y_1 , we have a constant K , depending only on the gradings of (a_1, \dots, a_ℓ) and \mathbf{x}_1 and \mathbf{y}_1 , so that

$$|w_i(b_1 \otimes \dots \otimes b_j)| \leq K.$$

(In more detail, consider the component of W_1 that matched $i \in Y_2$ with some $i' \in Y_1$, and let ξ_i and η_i be the coefficient of $\mathbf{gr}(\mathbf{x}_1)$ and $\mathbf{gr}(\mathbf{y}_1)$ in that component. Since X_1 is graded,

$$w_i(b_1 \otimes \dots \otimes b_j) + \eta_i = \xi_i + w_i(a_1 \otimes \dots \otimes a_\ell).$$

Now let $K = |\xi_i + w_i(a_1 \otimes \dots \otimes a_\ell) - \eta_i|$. By the same reasoning, we can adjust K so that, for any two points i and i' in Y_2 that are matched in W_1 ,

$$|w_i(b_1 \otimes \dots \otimes b_j) - w_{i'}(b_1 \otimes \dots \otimes b_j)| \leq K.$$

Suppose that $c \otimes \mathbf{y}_2$ appears with non-zero multiplicity in $\delta_{j+1}^1(\mathbf{x}_2, b_1, \dots, b_j)$, then for any point i' in Y_2 that is matched by W_2 to Y_3 , we can further adjust K (depending now on \mathbf{x}_2 and \mathbf{y}_2) so that

$$|w_i(b_1 \otimes \dots \otimes b_j) - w_{i'}(c)| \leq K.$$

By further adjusting K if necessary (depending only on \mathbf{x}_2 and \mathbf{y}_2), we can arrange that for any two points i and i' in Y_2 that are matched in W_2 ,

$$|w_i(b_1 \otimes \dots \otimes b_j) - w_{i'}(b_1, \dots, b_j)| \leq K.$$

Grading properties ensure that

$$(3.16) \quad \mathbf{m}(\mathbf{x}_1) + \mathbf{m}(\mathbf{x}_2) - \mathbf{m}(\mathbf{y}_1) - \mathbf{m}(\mathbf{y}_2) + \sum_{i=1}^{\ell} \mathbf{m}(a_i) = \mathbf{m}(c) + \ell - 1$$

By the above considerations, for any point i in Y_2 that is contained in a path connected component of $W_2 \cup W_1$ that meets Y_1 , there is a bound on $w_i(b_1 \otimes \dots \otimes b_j)$ depending only on $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$, and (a_1, \dots, a_ℓ) .

Suppose next that $i \in Y_2$ is contained in a component of $W_2 \cup W_1$ that meets Y_3 but not Y_1 . Consider the initial point p of that arc, with respect to the orientation it inherits from W , and observe that $p \in \mathcal{S}_3$. The above considerations give a bound on $|w_i(b_1 \otimes \dots \otimes b_j) - w_p(c)|$, again depending only on $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$, and (a_1, \dots, a_ℓ) . Finally, observe that since $p \in \mathcal{S}_3$, Equation (3.7) shows that we can adjust K so that (depending on $|\mathcal{S}_2|$) with

$$|\mathbf{m}(c) + 2w_p(c)| \leq K.$$

An upper bound on $\mathbf{m}(c)$ is provided by Equation (3.16). Thus, we have an upper bound on $w_i(b_1 \otimes \dots \otimes b_j)$ for any i in Y_2 that is contained in component of $W_2 \cup W_1$ that meets Y_3 but not Y_1 .

Since $W_2 \cup W_1$ has no closed components, we have obtained a universal bound on any weight $w_i(b_1 \otimes \dots \otimes b_j)$ for $i \in Y_2$. Now, for at least $j - \ell$ of the algebra

elements b_k , we have some i with $w_i(b_k) \geq 1/2$, so we have obtained a bound on j in terms of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$, and (a_1, \dots, a_ℓ) . This shows that the coefficient of $\mathbf{y}_1 \boxtimes \mathbf{y}_2$ in $\partial(\mathbf{x}_1 \boxtimes \mathbf{x}_2)$ is a sum of finitely many terms. Since $X^2 \boxtimes X^1$ is finitely generated, the stated finiteness follows. See Figure 18 \square

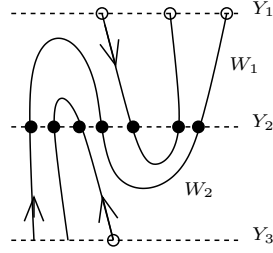


FIGURE 18. In the proof of Proposition 3.19, the total weights of the algebra elements at the circled points in Y_1 and Y_3 give bounds on the weights at points of Y_2 (drawn as dark dots) of the algebra elements in $b_1 \otimes \dots \otimes b_j$.

4. DD BIMODULES FOR CROSSINGS

Having defined the algebra associated to $y = t$ slice of the generic diagram, we turn now to the definitions of the modules associated to the partial knot diagrams. In Section 5, we will construct DA bimodules associated to special partial knot diagrams, consisting of a collection of vertical strands passing through the region $y_1 \leq y \leq y_2$ in the plane, containing exactly one crossing. Before doing this, we construct presently a simpler type DD bimodule associated to such a configuration.

4.1. The DD bimodule of a positive crossing. We describe first the DD bimodule \mathcal{P}_i associated to a positive crossing between the i^{th} and $(i+1)^{st}$ strands.

Let $\tau: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ be the transposition that switches i and $i+1$. Let

$$(4.1) \quad \mathcal{B}_1 = \mathcal{B}(m, k_1, \mathcal{S}_1) \quad \text{and} \quad \mathcal{B}_2 = \mathcal{B}(m, k_2, \mathcal{S}_2),$$

where $k_1 + k_2 = m + 1$, $|\mathcal{S}_1| + |\mathcal{S}_2| = m$, and $\mathcal{S}_1 \cap \tau(\mathcal{S}_2) = \emptyset$. We think of the algebra \mathcal{B}_1 as coming from below the crossing and algebra \mathcal{B}_2 as coming from above the crossing.

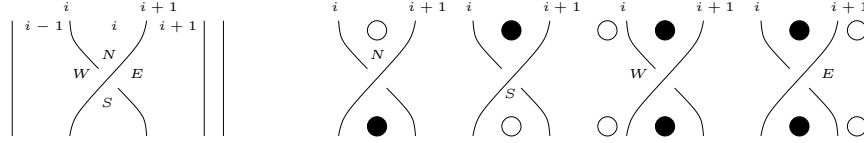


FIGURE 19. **Positive crossing DD bimodule generators.**
The four generator types are pictured to the right.

As an $\mathbf{I}(m, k_1, \mathcal{S}_1) - \mathbf{I}(m, k_2, \mathcal{S}_2)$ -bimodule, \mathcal{P}_i is the submodule of $\mathbf{I}(m, k_1, \mathcal{S}_1) \otimes_{\mathbb{F}} \mathbf{I}(m, k_2, \mathcal{S}_2)$ generated by elements $\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}$ where $\mathbf{x} \cap \mathbf{y} = \emptyset$ or

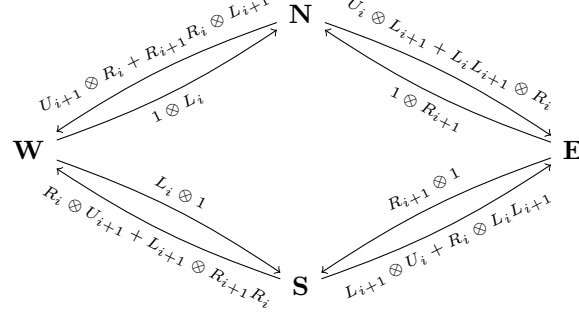
$$\mathbf{x} \cap \mathbf{y} = \{i\} \quad \text{and} \quad \{0, \dots, m\} \setminus (\mathbf{x} \cup \mathbf{y}) = \{i-1\} \text{ or } \{i+1\}.$$

In a little more detail, generators correspond to certain pairs of idempotent states \mathbf{x} and \mathbf{y} , where $|\mathbf{x}| = k_1$ and $|\mathbf{y}| = k_2$. They are further classified into four types, **N**, **S**, **W**, and **E**. For generators of type **N** the subsets \mathbf{x} and \mathbf{y} are complementary subsets of $\{0, \dots, m\}$ and $i \in \mathbf{x}$. For generators of type **S**, \mathbf{x} and \mathbf{y} are complementary subsets of $\{0, \dots, m\}$ with $i \in \mathbf{y}$. For generators of type **W**, $i-1 \notin \mathbf{x}$ and $i-1 \notin \mathbf{y}$, and $\mathbf{x} \cap \mathbf{y} = \{i\}$. For generators of type **E**, $i+1 \notin \mathbf{x}$ and $i+1 \notin \mathbf{y}$, and $\mathbf{x} \cap \mathbf{y} = \{i\}$.

The differential has the following types of terms:

- (P-1) $R_j \otimes L_j$ and $L_j \otimes R_j$ for all $j \in \{1, \dots, m\} \setminus \{i, i+1\}$; these connect generators of the same type.
- (P-2) $C_j \otimes U_{\tau(j)}$ if $j \in \mathcal{S}_1$ and $U_j \otimes C_{\tau(j)}$ if $j \notin \mathcal{S}_1$; these connect generators of the same type.

(P-3) Terms in the diagram below connect generators of different types:



(4.2)

Note that for a generator of type **E**, the terms of Type (P-1) with $j = i + 2$ vanish; while for one of type **W**, the terms of Type (P-1) with $j = i - 1$ vanish.

Proposition 4.1. *The bimodule $\mathcal{B}_1, \mathcal{B}_2 \mathcal{P}_i$ is a DD bimodule.*

Proof. The square of the differential, which we must verify vanishes, is obtained by either differentiating any of the terms of the Types (P-1)-(P-3) above or, multiplying together two of them.

We start by analyzing the terms of Types (P-3). Clearly, all of those terms have vanishing differential. The non-zero algebra elements obtained as products of pairs of such elements either connect generators of any type to itself, or it connects **N** and **S** or **W** and **E**.

As an example, consider products of terms of Type (P-3) connecting **W** to itself. Those products that factor through **N** give terms

$$(1 \otimes L_i) \cdot (U_{i+1} \otimes R_i + R_{i+1} R_i \otimes L_{i+1}) = U_{i+1} \otimes U_i + R_{i+1} R_i \otimes L_i L_{i+1}$$

and those that factor through **S** give

$$(L_i \otimes 1) \cdot (R_i \otimes U_{i+1} + L_{i+1} \otimes R_{i+1} R_i) = U_i \otimes U_{i+1} + L_i L_{i+1} \otimes R_{i+1} R_i.$$

Note that $(R_{i+1} R_i \otimes L_i L_{i+1} + L_i L_{i+1} \otimes R_{i+1} R_i) \otimes \mathbf{W}$ vanishes for idempotent reasons: if \mathbf{x} and \mathbf{y} are pairs of idempotent states with $(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) \cdot \mathbf{W} \neq 0$, then

$$|\mathbf{x} \cap \{i-1, i, i+1\}| + |\mathbf{y} \cap \{i-1, i, i+1\}| = 3.$$

The terms $U_i \otimes U_{i+1}$ and $U_{i+1} \otimes U_i$ cancel with the differentials of terms of Type (P-2) with $j = i$ and $i + 1$.

Next consider products of terms of Type (P-3) connecting **N** to **S**. The products that factor through **W** give terms:

$$(U_{i+1} \otimes R_i + R_{i+1} R_i \otimes L_{i+1}) \cdot (L_i \otimes 1) = L_i U_{i+1} \otimes R_i + R_{i+1} U_i \otimes L_{i+1}.$$

Similarly, the products that factor through **E** give

$$(U_i \otimes L_{i+1} + L_i L_{i+1} \otimes R_i) \cdot (R_{i+1} \otimes 1) = R_{i+1} U_i \otimes L_{i+1} + L_i U_{i+1} \otimes R_i;$$

thus, these two factorizations give terms that cancel in pairs.

The cancellation of terms of Type (P-3) in pairs or with differentials of terms of Type (P-2) with $j = i$ and $i + 1$ proceeds to other types of generators in the same way.

Next, we consider products of pairs of terms of Type (P-1). When $j \neq i, i+1$, these terms cancel differentials of terms of Type (P-2) exactly as in the proof Lemma 3.9. When the generators are of type **E** and **W**, there is a possible complication in this argument, since in that case, the corresponding term with $j = i+2$ or $i-1$ of Type (P-1) might vanish. For example, for terms of type **E**, and $i+2 \notin \mathcal{S}_1$, there is a possibly non-zero term $U_{i+2} \otimes C_{i+2}$ of Type (P-2), but the terms of Type (P-1) of the form $L_{i+2} \otimes R_{i+2}$ and $R_{i+2} \otimes L_{i+2}$ vanish. However, the differential of $U_{i+2} \otimes C_{i+2}$ vanishes in the idempotents of **E** (since if $\mathbf{I}_x \otimes \mathbf{I}_y$ is of type **E**, then $i+1 \notin x$ and $i+1 \notin y$, and either $i+2 \notin x$ or $i+2 \notin y$; in either case, $(U_{i+2} \otimes U_{i+2}) \cdot (\mathbf{I}_x \otimes \mathbf{I}_y) = 0$).

Consider next terms that are products of terms of Type (P-3) and those of Type (P-1). Typically, these are easily seen to cancel in pairs; the case where support of the Type (P-1) is immediately next to $\{i, i+1\}$ requires special care. Consider for example the terms of the form $L_{i+2} \otimes R_{i+2}$. Each term of Type (P-3) commutes with this term; but both product might be zero. For example, $L_i \otimes 1$ commutes with $(L_{i+2} \otimes R_{i+2})$. Also, if $a = L_i L_{i+1} \otimes R_i$, then $(L_{i+2} \otimes R_{i+2}) \cdot a = 0$ and $a \cdot (L_{i+2} \otimes R_{i+2}) = 0$.

The remaining terms of Type (P-2) and are easily seen to commute with each other and with terms of Type (P-3), giving the desired cancellation. \square

It is interesting to note that the bimodule has some symmetries; for instance,

$$(4.3) \quad \mathcal{B}_1, \mathcal{B}_2 \mathcal{P}_i \cong_{\mathcal{B}_2, \mathcal{B}_1} \mathcal{P}_i,$$

by a symmetry which switches the roles of **N** and **S**, and fixes **W** and **E**.

4.2. DD bimodule for a negative crossing. We can define a type *DD* bimodule for a negative crossing. The generators are the same as for a positive crossing. Terms in the differential are also the same, except that those of Type (P-3) are replaced by the following:

$$(4.4) \quad \begin{array}{ccccc} & & \mathbf{N} & & \\ & \swarrow & & \nwarrow & \\ \mathbf{W} & & & & \mathbf{E} \\ & \searrow & & \swarrow & \\ & & \mathbf{S} & & \end{array}$$

$\begin{array}{l} \text{Arrows from } \mathbf{W} \text{ to } \mathbf{N}: U_{i+1} \otimes L_i + L_i L_{i+1} \otimes R_{i+1} \\ \text{Arrows from } \mathbf{N} \text{ to } \mathbf{W}: 1 \otimes R_i \\ \text{Arrows from } \mathbf{W} \text{ to } \mathbf{S}: L_i \otimes U_{i+1} + R_{i+1} \otimes L_i L_{i+1} \\ \text{Arrows from } \mathbf{S} \text{ to } \mathbf{W}: R_i \otimes 1 \\ \text{Arrows from } \mathbf{N} \text{ to } \mathbf{E}: U_i \otimes R_{i+1} + R_{i+1} R_i \otimes L_i \\ \text{Arrows from } \mathbf{E} \text{ to } \mathbf{N}: 1 \otimes L_{i+1} \\ \text{Arrows from } \mathbf{E} \text{ to } \mathbf{S}: L_{i+1} \otimes 1 \\ \text{Arrows from } \mathbf{S} \text{ to } \mathbf{E}: R_{i+1} \otimes U_i + L_i \otimes R_{i+1} R_i \end{array}$

Proposition 4.2. *The bimodule \mathcal{N}_i is a DD bimodule.*

Proof. This follows exactly as in the proof of Proposition 4.1. Note that Diagram (4.4) is obtained from Diagram 4.2 by reversing all the arrows, and switching the roles of L_j and R_j . More formally, for \mathcal{B}_1 and \mathcal{B}_2 as in Equation (4.1),

$\mathcal{B}_1, \mathcal{B}_2 \mathcal{N}_i$ is obtained from the opposite module of \mathcal{P}_i , $\overline{\mathcal{P}}_i^{\mathcal{B}_1, \mathcal{B}_2} =_{\mathcal{B}_1^{\text{op}}, \mathcal{B}_2^{\text{op}}} \overline{\mathcal{P}}_i$ using the isomorphisms $\mathcal{B}_i \cong \mathcal{B}_i^{\text{op}}$ (denoted o in Equation (3.9)). \square

4.3. Gradings. The bimodules \mathcal{P}_i are graded by the set $S = \mathbb{Q}^m$ as follows. Let e_1, \dots, e_m be the standard basis for \mathbb{Q}^m . Let

$$(4.5) \quad \mathbf{gr}(\mathbf{N}) = \frac{e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \frac{e_i - e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \frac{-e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \frac{-e_i - e_{i+1}}{4},$$

Now, if $(a \otimes b) \otimes Y$ appears in ∂X , then

$$(4.6) \quad \mathbf{gr}(X) = \mathbf{gr}(a) - \tau^{\mathbf{gr}} \mathbf{gr}(b) + \mathbf{gr}(Y),$$

where $\tau_i^{\mathbf{gr}}$ is the linear transformation acting by τ_i on the standard basis vectors, and $\mathbf{gr}(a) = (w_1(a), \dots, w_m(a))$. Verifying Equation (4.6) is straightforward, using Equation (4.2).

In the notation of Section 2.2, the grading set of \mathcal{P}_i is half-integral valued functions on the arcs in the diagram, thought of as an affine space for $\Lambda_{\mathcal{B}_1} \times \Lambda_{\mathcal{B}_2}$ in an obvious way.

Similarly, for \mathcal{N}_i , the gradings of the four generators \mathbf{N} , \mathbf{S} , \mathbf{E} , \mathbf{W} are given by

$$(4.7) \quad \mathbf{gr}(\mathbf{N}) = \frac{-e_i - e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \frac{-e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \frac{e_i - e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \frac{e_i + e_{i+1}}{4},$$

Note that the modules determine these gradings only up to an overall additive shift. The present gradings are consistent with conventions on the multivariable Alexander polynomial [6]; for further motivation, see Remark 7.3.

5. DA BIMODULES ASSOCIATED TO CROSSINGS

Choose integers k and m with $0 \leq k \leq m+1$, and let $\mathcal{S} \subset \{1, \dots, m\}$ be arbitrary. Let $\tau_i: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ be the map that transposes i and $i+1$. Let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S})$ and $\mathcal{B}_2 = \mathcal{B}(m, k, \tau(\mathcal{S}))$. The aim of the present section is to construct a type DA bimodule ${}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i$, which we think of as the bimodule associated to a region in the knot diagram $t_2 \leq y \leq t_1$ that contains exactly one “positive” crossing, and no local maxima or minima; and the crossing occurs between the i^{th} and $(i+1)^{\text{st}}$ strands, as shown on the left in Figure 20. (Positivity here is meant with respect to a braid orientation, e.g. where all the strands are oriented upwards, which might differ from the orientation specified by \mathcal{S} .)

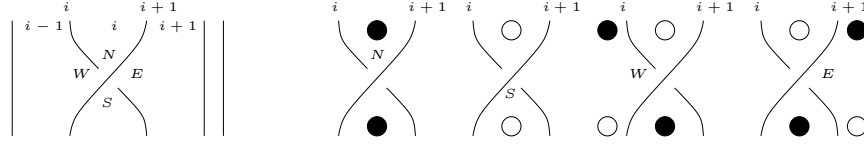


FIGURE 20. **Positive crossing DA bimodule generators.**
The four generator types are pictured to the right.

Consider the submodule \mathcal{P}^i of $\mathbf{I}(m, k) \otimes_{\mathbb{F}} \mathbf{I}(m, k)$, consisting of $\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}$ where either $\mathbf{x} = \mathbf{y}$ or there is some $\mathbf{w} \subset \{1, \dots, i-1, i+1, \dots, m\}$ with $\mathbf{x} = \mathbf{w} \cup \{i\}$ and $\mathbf{y} = \mathbf{w} \cup \{i-1\}$ or $\mathbf{y} = \mathbf{w} \cup \{i+1\}$. Thus, there are once again four types of generators, of type **N**, **S**, **W**, **E** as pictured in Figure 20; i.e.

$$\begin{aligned} \sum_{i \in \mathbf{x}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{N} \cdot \mathbf{I}_{\mathbf{x}} &= \mathbf{N}, & \sum_{i \notin \mathbf{x}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{S} \cdot \mathbf{I}_{\mathbf{x}} &= \mathbf{S}, \\ \sum_{\substack{i \in \mathbf{x} \\ i-1 \notin \mathbf{x}}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{W} \cdot \mathbf{I}_{\{i-1\} \cup \mathbf{x} \setminus \{i\}} &= \mathbf{W}, & \sum_{\substack{i \in \mathbf{x} \\ i+1 \notin \mathbf{x}}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{E} \cdot \mathbf{I}_{\{i+1\} \cup \mathbf{x} \setminus \{i\}} &= \mathbf{E}. \end{aligned}$$

This description has a geometric interpretation in terms of Kauffman states.

Definition 5.1. A partial knot diagram is the portion of a knot diagram contained in the (x, y) plane with $t_2 \leq y \leq t_1$, so that the diagram meets the slices $y = t_1$ and $y = t_2$ generically. The knot projection divides the partial knot diagram into regions. A partial Kauffman state is a triple of data $(\mathbf{K}, \mathbf{x}, \mathbf{y})$, where \mathbf{K} is a map that associates to each crossing one of its four adjacent regions; \mathbf{y} is a collection of intervals in the intersection of the diagram with the $y = t_1$ slice; and \mathbf{x} is a collection of intervals in the intersection of the diagram with the $y = t_2$ slice. The regions assigned to the crossings are called occupied regions; those that are not are called unoccupied. The data in a partial Kauffman state are further required to satisfy the following compatibility conditions:

- No two crossings are assigned to the same region.
- If a region R is occupied, then \mathbf{y} contains all the intervals in $R \cap (y = t_1)$ and \mathbf{x} contains none of the regions in $R \cap (y = t_2)$.
- If a region R is unoccupied, then either \mathbf{y} contains all but one of the regions in $R \cap (y = t_1)$ and \mathbf{x} contains none of the regions in $R \cap (y = t_2)$; or \mathbf{y}

contains all of the regions in $R \cap (y = t_1)$ and the region \mathbf{x} contains exactly one of the regions of $R \cap (y = t_2)$

The picture is simplified considerably when the partial knot diagram contains a single crossing. In that case, it is straightforward to see that the generators of the bimodule \mathcal{P}^i defined above correspond to partial Kauffman states; \mathbf{K} is one of \mathbf{N} , \mathbf{S} , \mathbf{W} , or \mathbf{E} ; \mathbf{y} specifies the right idempotent of the generator and \mathbf{x} the left idempotent of the generator.

The bimodules \mathcal{P}^i are graded by the set \mathbb{Q}^m in the following sense. Define gradings of the generators \mathbf{N} , \mathbf{S} , \mathbf{E} , and \mathbf{W} as in Equation (4.5) (only now thinking of these as generators of \mathcal{P}^i rather than \mathcal{P}_i). The actions δ_ℓ^1 respect this grading, in the following sense. Suppose that X is some homogeneous generator, and $a_1, \dots, a_\ell \in \mathcal{B}_1$ are homogeneous elements, then $\delta_\ell^1(X, a_1, \dots, a_\ell)$ can be written as a sum of elements of the form $b \otimes Y$ where Y are homogeneous generators and $b \in \mathcal{B}_2$ is a homogeneous element of the algebra, with

$$(5.1) \quad \mathbf{gr}(X) + \tau_i^{\mathbf{gr}}(\mathbf{gr}(a_1) + \dots + \mathbf{gr}(a_{\ell-1})) = \mathbf{gr}(b) + \mathbf{gr}(Y),$$

where here $\mathbf{gr}(a_i)$ and $\mathbf{gr}(b)$ denote the weight gradings on \mathcal{B}_1 and \mathcal{B}_2 .

There is also a \mathbb{Z} -grading, specified by where the grading of a generator type coincides with the local Maslov contribution of the corresponding Kauffman state as in Figure 2. For instance, if all crossings are oriented upwards, then

$$(5.2) \quad M(\mathbf{N}) = -1 \quad M(\mathbf{S}) = M(\mathbf{W}) = M(\mathbf{E}) = 0.$$

The modules respect this \mathbb{Z} -grading in the sense that, for X , Y , $a_1, \dots, a_{\ell-1}$ and b as above,

$$(5.3) \quad M(X) + M(a_1) + \dots + M(a_{\ell-1}) + \ell - 1 = M(b) + M(Y).$$

The partial knot diagram with a single crossing in it determines in an obvious way a collection of arcs. Equation (5.1) can be interpreted as saying that the bimodule \mathcal{P}^i is adapted to this one-manifold with boundary, in the sense of Definition 3.18.

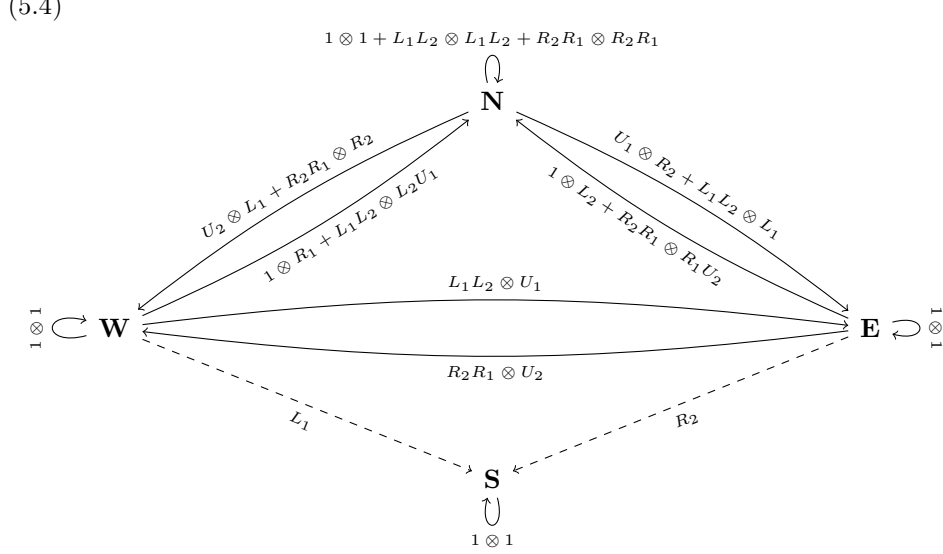
We consider first the case where $\mathcal{S} \cap \{i, i+1\} = \emptyset$, we will specify these actions by localizing to the crossing region, defining first bimodules over $\mathcal{B}(2) = \mathcal{B}(2, 0) \oplus \mathcal{B}(2, 1) \oplus \mathcal{B}(2, 2) \oplus \mathcal{B}(2, 3)$, and then extending them to $\mathcal{B}(m, k, \mathcal{S})$ (with $\mathcal{S} \cap \{i, i+1\} = \emptyset$) in Subsection 5.2. The local models when $\mathcal{S} \cap \{i, i+1\}$ is non-empty is defined in Section 5.3, and it is extended in the general case in 5.4.

5.1. Local bimodule for a positive crossing. We define first a type DA bimodule P over $\mathcal{B}(2) = \mathcal{B}(2, 0) \oplus \mathcal{B}(2, 1) \oplus \mathcal{B}(2, 2) \oplus \mathcal{B}(2, 3)$. Note that $\mathcal{B}(2)$ is defined over $\mathbf{I}(2) = \mathbf{I}(2, 0) \oplus \mathbf{I}(2, 1) \oplus \mathbf{I}(2, 2) \oplus \mathbf{I}(2, 3) \cong \mathbb{F}^8$.

As an $\mathbf{I}(2) - \mathbf{I}(2)$ -module, P has four generators \mathbf{N} , \mathbf{S} , \mathbf{W} , and \mathbf{E} . As an \mathbb{F} -vector space, P is 12 dimensional, and all 12 basis vectors appear on the right hand sides of the following four expressions, which determine the $\mathbf{I}(2) - \mathbf{I}(2)$ -module structure:

$$\begin{aligned} \mathbf{N} &= \mathbf{I}_{\{1\}} \cdot \mathbf{N} \cdot \mathbf{I}_{\{1\}} + \mathbf{I}_{\{0,1\}} \cdot \mathbf{N} \cdot \mathbf{I}_{\{0,1\}} + \mathbf{I}_{\{1,2\}} \cdot \mathbf{N} \cdot \mathbf{I}_{\{1,2\}} + \mathbf{I}_{\{1,2,3\}} \cdot \mathbf{N} \cdot \mathbf{I}_{\{1,2,3\}} \\ \mathbf{S} &= \mathbf{I}_\emptyset \cdot \mathbf{S} \cdot \mathbf{I}_\emptyset + \mathbf{I}_{\{0\}} \cdot \mathbf{S} \cdot \mathbf{I}_{\{0\}} + \mathbf{I}_{\{2\}} \cdot \mathbf{S} \cdot \mathbf{I}_{\{2\}} + \mathbf{I}_{\{0,2\}} \cdot \mathbf{S} \cdot \mathbf{I}_{\{0,2\}} \\ \mathbf{W} &= \mathbf{I}_{\{1\}} \cdot \mathbf{W} \cdot \mathbf{I}_{\{0\}} + \mathbf{I}_{\{1,2\}} \cdot \mathbf{W} \cdot \mathbf{I}_{\{0,2\}} \\ \mathbf{E} &= \mathbf{I}_{\{1\}} \cdot \mathbf{E} \cdot \mathbf{I}_{\{2\}} + \mathbf{I}_{\{0,1\}} \cdot \mathbf{E} \cdot \mathbf{I}_{\{0,2\}}. \end{aligned}$$

Next, we define the δ_1^1 and δ_2^1 actions on P . Some of these actions are specified in the following diagram:



Here, dashed arrows indicate δ_1^1 actions. For example, $\delta_1^1(\mathbf{W}) = L_1 \otimes \mathbf{S}$.

Further actions are obtained by the following *local extension rules*. For any $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}, \mathbf{S}\}$ and any pure algebra element $a \in \mathcal{B}(2)$,

$$(5.5) \quad \delta_2^1(X, U_1 U_2 \cdot a) = U_1 U_2 \cdot \delta_2^1(X, a).$$

and also:

- If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{N}, a)$, then $(b \cdot U_2) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{N}, a \cdot U_1)$ and $(b \cdot U_1) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{N}, a \cdot U_2)$.
- If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{W}, a)$, then $(U_2 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{W}, U_1 \cdot a)$.
- If $b \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{E}, a)$, then $(U_1 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta_2^1(\mathbf{E}, U_2 \cdot a)$.

For example, the action $\delta_2^1(\mathbf{W}, 1) = \mathbf{W}$ combined with the local extension rules shows that

$$\delta_2^1(\mathbf{W}, U_1) = U_2 \otimes \mathbf{W} + L_1 L_2 \otimes \mathbf{E}.$$

The above rules uniquely specify the \mathbb{F} -linear map $\delta_2^1: \mathcal{P} \otimes \mathcal{B}(2) \rightarrow \mathcal{B}(2) \otimes \mathcal{P}(2)$.

We wish next to specify δ_3^1 . As in the case for δ_2^1 , we define these on a basis. More formally, an algebra element is called *elementary* if it is of the form $p \cdot e$, where p is a monomial in U_1 and U_2 , and

$$e \in \{1, \quad L_1, \quad R_1, \quad L_2, \quad R_2, \quad L_1 L_2, \quad R_2 R_1\}.$$

We have defined δ_2^1 by specifying the δ_2^1 actions of the form $\delta_2^1(X, a)$, where $X \in \{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$ and a is elementary. We will specify $\delta_3^1(X, a_1, a_2)$ where a_1 and a_2 are elementary.

Suppose that a_1 and a_2 are elementary algebra elements with $a_1 \otimes a_2 \neq 0$ (i.e. there is an idempotent state \mathbf{x} so that $a_1 \cdot \mathbf{I}_{\mathbf{x}} \neq 0$ and $\mathbf{I}_{\mathbf{x}} \cdot a_2 \neq 0$); and suppose moreover that $U_1 \cdot U_2$ does not divide either a_1 nor a_2 . In this case, $\delta_3^1(\mathbf{S}, a_1, a_2)$ is the sum of terms:

- $R_1 U_1^t \otimes \mathbf{E}$ if $(a_1, a_2) = (R_1, R_2 U_2^t)$ and $t \geq 0$
- $L_2 U_1^t U_2^n \otimes \mathbf{E}$ if $(a_1, a_2) \in$

$$\begin{aligned} &\{(U_1^{n+1}, U_2^t), (R_1 U_1^n, L_1 U_2^t), (L_2 U_1^{n+1}, R_2 U_2^{t-1})\} \quad \text{when } 0 \leq n < t \\ &\{(U_2^t, U_1^{n+1}), (R_1 U_2^t, L_1 U_1^n), (L_2 U_2^{t-1}, R_2 U_1^{n+1})\} \quad \text{when } 1 \leq t \leq n \end{aligned}$$
- $L_2 U_2^n \otimes \mathbf{W}$ if $(a_1, a_2) = (L_2, L_1 U_1^n)$ and $n \geq 0$
- $R_1 U_1^t U_2^n \otimes \mathbf{W}$ if $(a_1, a_2) \in$

$$\begin{aligned} &\{(U_2^{t+1}, U_1^n), (L_2 U_2^t, R_2 U_1^n), (R_1 U_2^{t+1}, L_1 U_1^{n-1})\} \quad \text{when } 0 \leq t < n \\ &\{(U_1^n, U_2^{t+1}), (L_2 U_1^n, R_2 U_2^t), (R_1 U_1^{n-1}, L_1 U_2^{t+1})\}, \quad \text{when } 1 \leq n \leq t \end{aligned}$$
- $L_2 U_1^t U_2^n \otimes \mathbf{N}$ if $(a_1, a_2) \in$

$$\begin{aligned} &\{(U_1^{n+1}, L_2 U_2^t), (R_1 U_1^n, L_1 L_2 U_2^t), (L_2 U_1^{n+1}, U_2^t)\} \quad \text{when } 0 \leq n < t \\ &\{(L_2 U_2^t, U_1^{n+1}), (U_2^t, L_2 U_1^{n+1}), (R_1 U_2^t, L_1 L_2 U_1^n)\} \quad \text{when } 1 \leq t \leq n \\ &\{(L_2, U_1^{n+1})\} \quad \text{when } 0 = t \leq n \end{aligned}$$
- $R_1 U_1^t U_2^n \otimes \mathbf{N}$ if (a_1, a_2) is in the following list:
$$\begin{aligned} &\{(U_2^{t+1}, R_1 U_1^n), (L_2 U_2^t, R_2 R_1 U_1^n), (R_1 U_2^{t+1}, U_1^n)\} \quad \text{when } 0 \leq t < n \\ &\{(R_1 U_1^n, U_2^{t+1}), (U_1^n, R_1 U_2^{t+1}), (L_2 U_1^n, R_2 R_1 U_2^t)\} \quad \text{when } 1 \leq n \leq t \\ &\{(R_1, U_2^{t+1})\} \quad \text{when } 0 = n \leq t. \end{aligned}$$

For example

$$\delta_3^1(\mathbf{S}, U_1, U_2^2) = (L_2 U_1^2) \otimes \mathbf{E} + (R_1 U_1 U_2) \otimes \mathbf{W}.$$

Extend this to the case where $U_1 U_2$ divides a_1 or a_2 or both, by requiring

$$(5.6) \quad \delta_3^1(Z, (U_1 U_2) \cdot a, b) = \delta_3^1(Z, a, (U_1 U_2) \cdot b) = (U_1 U_2) \cdot \delta_3^1(Z, a, b).$$

Proposition 5.2. *The operations δ_ℓ^1 defined above give P the structure of a DA bimodule over $\mathcal{B}(2) = \mathcal{B}(2, 0) \oplus \mathcal{B}(2, 1) \oplus \mathcal{B}(2, 2) \oplus \mathcal{B}(2, 3)$, which is graded as in Equations (5.1) and (5.3).*

We prove the above proposition after some lemmas.

Lemma 5.3. *The actions δ_ℓ^1 respect the gradings, as in Equation (5.1).*

Proof. In the present case, the gradings on \mathcal{P}^i are specified by

$$\mathbf{gr}(\mathbf{N}) = \frac{e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \frac{e_i - e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \frac{-e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \frac{-e_i - e_{i+1}}{4},$$

Bearing this in mind, the lemma is a straightforward verification using the above defined actions δ_ℓ^1 . \square

Observe that if $b \otimes Y$ appears with non-zero multiplicity in $\delta_\ell^1(X, a_1, \dots, a_{\ell-1})$, and X is a fixed generator, and the $a_i \in \mathcal{B}(2)$ are elementary for $i = 1, \dots, \ell - 1$, then the output algebra b is uniquely specified in Equation (5.1).

Lemma 5.4. *Given a monomial p in U_1 and U_2 and an algebra element*

$$a \in \{p, \quad R_1 \cdot p, \quad L_2 \cdot p, \quad R_2 R_1 \cdot p, \quad L_1 L_2 \cdot p\},$$

there is a unique $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ and b so that $b \otimes \mathbf{N}$ appears with non-zero multiplicity in $\delta_2^1(X, a)$. Similarly, given a monomial p in U_1 and U_2 an algebra element from

$$a \in \{p, \quad L_1 \cdot p, \quad R_2 \cdot p\},$$

and a fixed state type $Y \in \{\mathbf{W}, \mathbf{E}\}$, there is a unique state type $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ and b so that $b \otimes Y$ appears with non-zero multiplicity in $\delta_2^1(X, a)$.

Proof. This follows from a straightforward inspection of Diagram (5.4) and the local extension rules. For example, if $Y = \mathbf{N}$, then if $a = p \cdot e$ where p is a monomial in U_1 and U_2 , and $e \in \{1, L_1 L_2, R_2 R_1\}$, then $X = \mathbf{N}$. For $e = L_2$, write $p = U_1^x U_2^y$. If $x > y$, then $X = \mathbf{W}$, and if $x \leq y$, then $X = \mathbf{E}$. The case where $e = R_1$ works similarly. \square

For any elementary a and $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ with $a \otimes Y \neq 0$, define $I(a, Y)$ to be the generator $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ as defined in Lemma 5.4. Otherwise (i.e. if the pair (a, Y) is not covered by the lists of Lemma 5.4), define $I(a, Y) = 0$.

Lemma 5.5. *Fix elementary algebra elements a_1, a_2 in $\mathcal{B}(2)$, with $\mathbf{S} \otimes a_1 \otimes a_2 \neq 0$, an elementary b and $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ so that $b \otimes Y \neq 0$ and*

$$(5.7) \quad \mathbf{gr}(\mathbf{S} \otimes a_1 \otimes a_2) = \mathbf{gr}(b \otimes Y).$$

Suppose moreover that $a_1 \cdot a_2 \neq 0$. Then the element $b \otimes Y$ appears with non-zero multiplicity in $\delta_3^1(\mathbf{S}, a_1, a_2)$ if and only if $I(a_2, Y) \neq 0$ and $I(a_1, I(a_2, Y)) \neq I(a_1 \cdot a_2, Y)$; (i.e. one of them is \mathbf{E} and the other is \mathbf{W}).

Proof. This follows from a straightforward inspection of the definition of δ_3^1 . \square

We introduce a notational shorthand. The maps

$$\delta_i^1: P \otimes \overbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}^{i-1} \rightarrow \mathcal{B} \otimes P$$

naturally extend to maps

$$\tilde{\delta}_i^1: \mathcal{B} \otimes P \otimes \overbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}^{i-1} \rightarrow \mathcal{B} \otimes P,$$

by the rule

$$\delta_i^1(b \otimes x \otimes a_1 \otimes \cdots \otimes a_{i-1}) = b \cdot \delta_i^1(x \otimes a_1 \otimes \cdots \otimes a_{i-1}).$$

Fix elementary algebra elements $a_1, a_2 \in \mathcal{B}(2, k)$, so that $\mathbf{S} \otimes a_1 \otimes a_2 \neq 0$. When $k = 2$, $a_1 \cdot a_2$ is always non-zero. On the other hand, when $k = 1$, the actions $\delta_3^1(\mathbf{S}, a_1, a_2)$ are those for which $a_1 \cdot a_2 = 0$ and $b \otimes Y$ (as specified by Equation (5.7)) appears with non-zero multiplicity in $\tilde{\delta}_2^1(\delta_2^1(\mathbf{S}, a_1), a_2)$.

Lemma 5.6. *With the above definition, if a_1, a_2 , and a_3 are elementary, then*

$$(5.8) \quad \begin{aligned} & \tilde{\delta}_2^1(\delta_3^1(\mathbf{S}, a_1, a_2), a_3) + \tilde{\delta}_3^1(\delta_2^1(\mathbf{S}, a_1), a_2, a_3) \\ & + \delta_3^1(\mathbf{S}, a_1 \cdot a_2, a_3) + \delta_3^1(\mathbf{S}, a_1, a_2 \cdot a_3) = 0 \end{aligned}$$

Proof. The above actions δ_3^1 defined above vanish on $\mathcal{B}(2, 0)$ and $\mathcal{B}(2, 3)$, so Equation (5.8) is obvious. On $\mathcal{B}(2, 1)$, the non-trivial δ_3^1 actions are

$$\begin{aligned}\delta_3^1(\mathbf{S}, R_1, R_2 U_2^t) &= R_1 U_1^t \otimes \mathbf{E} \\ \delta_3^1(\mathbf{S}, L_2, L_1 U_1^n) &= L_2 U_2^n \otimes \mathbf{W} \\ \delta_3^1(\mathbf{S}, L_2, U_1^{n+1}) &= L_2 U_2^n \otimes \mathbf{N} \\ \delta_3^1(\mathbf{S}, R_1, U_2^{t+1}) &= R_1 U_1^t \otimes \mathbf{N}\end{aligned}$$

with $k, n \geq 0$. For these, the verification of Equation (5.8) is straightforward.

In $\mathcal{B}(2, 2)$, if $a_1 \otimes a_2 \otimes a_3 \neq 0$, then in fact $a_1 \cdot a_2 \cdot a_3 \neq 0$; so we will henceforth assume that $a_1 \cdot a_2 \cdot a_3 \neq 0$. We treat two subcases separately, according to whether or not $a_1 = (U_1 U_2)^\ell$.

Case 1: $a_1 = (U_1 U_2)^\ell$. The first and the fourth terms in Equation (5.8) vanish, and the middle two agree by Equation (5.6).

Case 2: $a_1 \neq (U_1 U_2)^\ell$. In this case, $\delta_2^1(\mathbf{S}, a_1) = 0$, so the second term in Equation (5.8) is missing. Given $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$, let

$$\begin{aligned}\alpha &= I(a_1 \cdot a_2, I(a_3, Y)), & \beta &= I(a_1 \cdot a_2 \cdot a_3, Y), \\ \gamma &= I(a_1, I(a_2 \cdot a_3, Y)), & \delta &= I(a_1, I(a_2, I(a_3, Y))).\end{aligned}$$

By Lemma 5.5, $b \otimes Y$ (where $\mathbf{gr}(\mathbf{S} \otimes a_1 \otimes a_2 \otimes a_3) = \mathbf{gr}(b \otimes Y)$) appears with non-zero multiplicity in:

- $\delta_3^1(\mathbf{S}, a_1 \cdot a_2, a_3)$ if the set $\{\alpha, \beta\}$ equals $\{\mathbf{W}, \mathbf{E}\}$
- $\delta_3^1(\mathbf{S}, a_1, a_2 \cdot a_3)$ if $\{\gamma, \beta\} = \{\mathbf{W}, \mathbf{E}\}$
- $\tilde{\delta}_2^1(\delta_3^1(\mathbf{S}, a_1, a_2), a_3)$ if $\{\alpha, \delta\} = \{\mathbf{W}, \mathbf{E}\}$

Equation 5.8 states that either $\alpha = \beta = \gamma = \delta$ or exactly one of the three relations ($\alpha = \beta, \beta = \gamma, \alpha = \delta$) holds. To establish this, we will need the following two observations about $I(a, Z)$:

Observation 1. If a is elementary and Z is a generator type, then $I(a, Z)$ is uniquely determined by a and whether or not $Z = \mathbf{N}$, except in the special case where $a = (U_1 U_2)^\ell$ for some $\ell \geq 0$. For example, if $a = L_1 \cdot p$ or $R_2 \cdot p$, where p is a monomial in U_1 and U_2 , then $I(a, Z) = \mathbf{N}$; if $a = U_1^x \cdot U_2^y$ and $Z \in \{\mathbf{W}, \mathbf{E}\}$, then $I(a, Z) = \mathbf{W}$ if $x > y$ and $I(a, Z) = \mathbf{E}$ if $x < y$.

The generator types $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ are further grouped into two classes, $\{\mathbf{N}\}$ and $\{\mathbf{W}, \mathbf{E}\}$; in this language, the Observation 1 can be phrased as saying that if $a \neq (U_1 U_2)^\ell$, the class of Z and the algebra element a uniquely determines $I(a, Z)$.

Observation 2. If a_1 and a_2 are elementary and $a_1 \cdot a_2 \neq 0$, then the class of $I(a_1, I(a_2, Z))$ agrees with the class of $I(a_1 \cdot a_2, Z)$.

By the second observation, the class of $I(a_2 \cdot a_3, Y)$ agrees with the class of $I(a_2, I(a_3, Y))$, so by the first observation, $I(a_1, I(a_2 \cdot a_3, Y)) = I(a_1, I(a_2, I(a_3, Y)))$; i.e. $\gamma = \delta$, so if any two of the conditions ($\alpha = \beta, \beta = \gamma, \alpha = \delta$) holds, the third one holds, verifying Equation (5.8) when $a_1 \neq (U_1 U_2)^\ell$. \square

Proof of Proposition 5.2. Note that in $\mathcal{B}(2, 3)$, there is only one non-trivial generator, and it is \mathbf{N} ; so $\delta_\ell^1 = 0$ for all $\ell > 2$. In fact, \mathcal{P}^1 in this case is

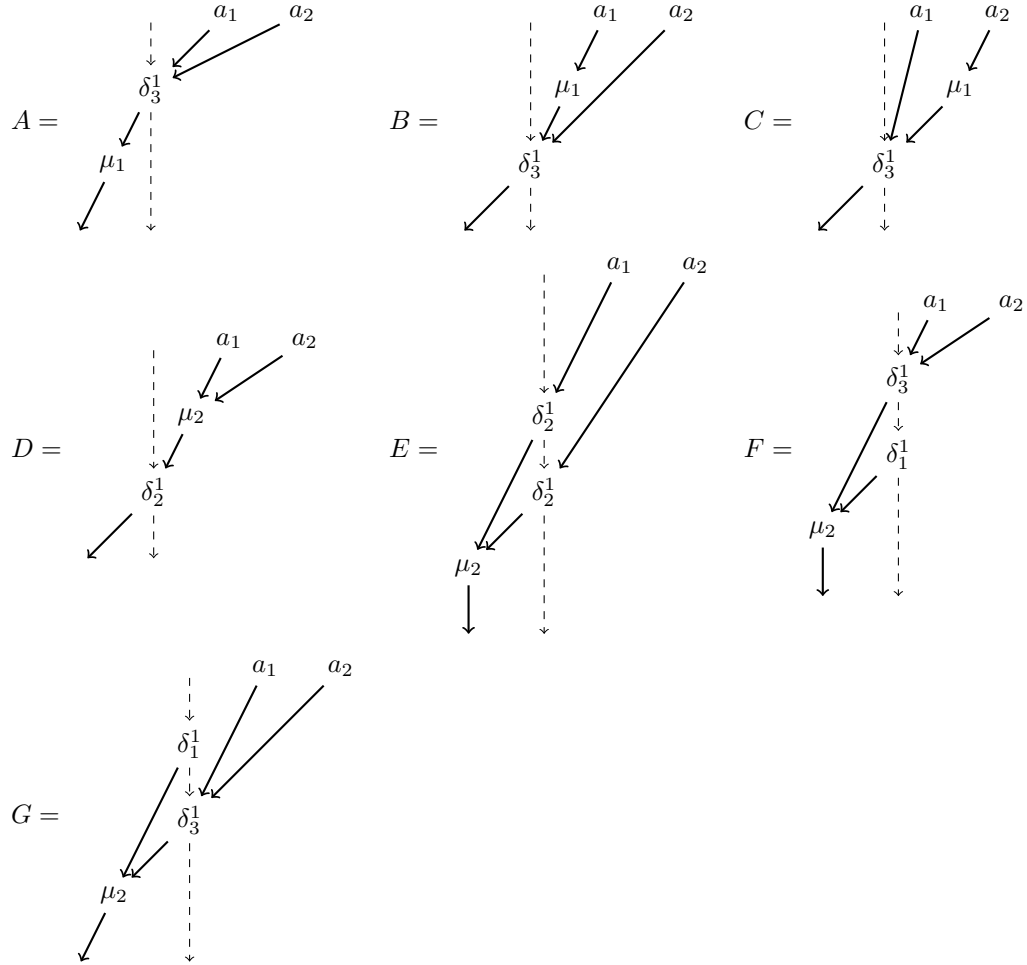
simply the identity bimodule for $\mathcal{B}(2,3)$, and the \mathcal{A}_∞ relation holds. A similar simplification occurs for $\mathcal{B}(2,0)$, where the algebra is one-dimensional, and \mathbf{S} is the only module generator.

We turn to the more interesting cases, in the summands $\mathcal{B}(2,1)$ and $\mathcal{B}(2,2)$. Varying n , we verify the DA bimodule relation with n algebra inputs a_1, \dots, a_n (and one module input), subdividing further each verification according to whether we are in $\mathcal{B}(2,1)$ or $\mathcal{B}(2,2)$.

When $n = 0$, the \mathcal{A}_∞ relation is clear from the form of δ_1^1 .

When $n = 1$, terms of the form $\tilde{\delta}_1^1 \circ \delta_2^1$ cancel in pairs, except in the special case where $a_1 = (U_1 U_2)^\ell$, in which case a term of the form $\tilde{\delta}_1^1 \circ \delta_2^1$ cancels with another of the form $\tilde{\delta}_2^1 \circ \delta_1^1$.

When $n = 2$, we find it convenient to label the terms in the \mathcal{A}_∞ relation as follows:
(5.9)



So that the \mathcal{A}_∞ relation on a DA bimodule with two inputs reads

$$A + B + C + D + E + F + G = 0.$$

On the algebras we are considering presently, $\mu_1 = 0$, so $A = B = C = 0$.

When $n = 2$, since all actions are U_1U_2 -equivariant, the case where at least one of a_1 or a_2 is $(U_1U_2)^\ell$ is straightforward: the only two possibly non-zero terms are D and E , and they cancel.

When $n = 2$, and we are in the summand in $\mathcal{B}(2, 2)$, and the starting and ending generator is in $\{\mathbf{E}, \mathbf{W}, \mathbf{N}\}$, the \mathcal{A}_∞ relation follows from Lemma 5.5, and the observation that $a_1 \cdot a_2 \neq 0$. Specifically, in this case, $A = B = C = 0$, and the δ_3^1 actions are defined so that the term in G cancels D and E .

When $n = 2$ and we are in the summand in $\mathcal{B}(2, 2)$, and the starting and ending generator is \mathbf{S} , then $A = B = C = G = 0$, and we find that terms of type F cancel in pairs, except in the special cases where $a_1 \cdot a_2 = (U_1U_2)^\ell$, which are the cases where $D \neq 0$, in which case there is a single cancelling term of type F . This is verified by looking at the formulas defining δ_3^1 . For example, consider an \mathcal{A}_∞ relation with generator of type \mathbf{S} , $a_1 = U_1^n$, and $a_2 = U_2^t$. If $1 \leq n \leq t$, there is a term of type F , factoring through \mathbf{E} , and this cancels against another term of type D , factoring through \mathbf{W} , when $n < t$; otherwise, it cancels against a contribution of type D .

When $n = 2$ and we are in the summand in $\mathcal{B}(2, 1)$ and the initial generator is of type $\{\mathbf{E}, \mathbf{W}, \mathbf{N}\}$, terms of type D and E cancel except in special cases where E contributes but $a_1 \cdot a_2 = 0$, in which case there is a cancelling non-zero term of type G . When the initial generator is of type \mathbf{S} , and neither of a_1 nor a_2 equals $(U_1U_2)^\ell$, the only possible non-zero term, which is of type F , vanishes thanks to the algebra; for example, in the \mathcal{A}_∞ relation $\delta_3^1(\mathbf{S}, R_1, R_2U_2^t) = R_1U_1^t \otimes \mathbf{E}$, and $\delta_1^1(\mathbf{E}) = R_2 \otimes \mathbf{S}$, but $R_1 \cdot R_2 = 0$ in $\mathcal{B}(2, 1)$.

The cases where $n = 3$, the \mathcal{A}_∞ relation trivially holds except in the special cases where the initial generator is of type \mathbf{S} . That case was covered by Lemma 5.6.

The case where $n > 3$ is obvious. \square

5.2. Extension. Fix integers k and m with $0 \leq k \leq m + 1$, we extend the bimodule P to a bimodule ${}^{\mathcal{B}(m,k)}\mathcal{P}_{\mathcal{B}(m,k)}^i$, as follows.

Let $a \in \mathcal{B}_0(m, k)$ with $a = \mathbf{I}_\mathbf{x} \cdot a \cdot \mathbf{I}_\mathbf{y}$, and suppose that \mathbf{x} and \mathbf{y} are close enough. Suppose moreover that a is pure, in the sense that it corresponds to some monomial in U_1, \dots, U_m under $\phi^{\mathbf{x}, \mathbf{y}}$. We define the *type* of a , denoted $t(a)$, which is an

expression in in U_1, U_2, R_1, L_1, R_2 and L_2 , defined as follows:

$$t(a) = \begin{cases} R_2 R_1 U_1^{w_i(a)-\frac{1}{2}} U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \\ & \text{and } v_{i+1}^{\mathbf{x}} < v_{i+1}^{\mathbf{y}} \\ L_1 L_2 U_1^{w_i(a)-\frac{1}{2}} U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \\ & \text{and } v_{i+1}^{\mathbf{x}} > v_{i+1}^{\mathbf{y}} \\ R_2 U_1^{w_i(a)} U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \in \mathbb{Z} \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}}, \\ & \text{and } v_{i+1}^{\mathbf{x}} < v_{i+1}^{\mathbf{y}} \\ L_2 U_1^{w_i(a)} U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \in \mathbb{Z} \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}}, \\ & \text{and } v_{i+1}^{\mathbf{x}} > v_{i+1}^{\mathbf{y}} \\ R_1 U_1^{w_i(a)-\frac{1}{2}} U_2^{w_{i+1}(a)} & \text{if } w_i(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } w_{i+1}(a) \in \mathbb{Z}, \\ & \text{and } v_i^{\mathbf{x}} < v_i^{\mathbf{y}} \\ L_1 U_1^{w_i(a)-\frac{1}{2}} U_2^{w_{i+1}(a)} & \text{if } w_i(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } w_{i+1}(a) \in \mathbb{Z}, \\ & \text{and } v_i^{\mathbf{x}} > v_i^{\mathbf{y}} \\ U_1^{w_i(a)} U_2^{w_{i+1}(a)} & \text{if } w_i(a) \text{ and } w_{i+1}(a) \text{ are integers.} \end{cases}$$

Similarly, there is a map t from generators of \mathcal{P}^i to the four generators of P , that remembers only the type (**N**, **S**, **W**, **E**) of the generator of \mathcal{P}^i .

Definition 5.7. For $X \in \mathcal{P}^i$, an integer $\ell \geq 1$, and a sequence of algebra elements $a_1, \dots, a_{\ell-1}$ in $\mathcal{B}_0(m, k)$ with specified weights, so that there exists a sequence of idempotent states $\mathbf{x}_0, \dots, \mathbf{x}_\ell$ with

- $X = \mathbf{I}_{\mathbf{x}_0} \cdot X \cdot \mathbf{I}_{\mathbf{x}_1}$
- $a_t = \mathbf{I}_{\mathbf{x}_t} \cdot a_t \cdot \mathbf{I}_{\mathbf{x}_{t+1}}$ for $t = 1, \dots, \ell - 1$
- \mathbf{x}_t and \mathbf{x}_{t+1} are close enough (for $t = 0, \dots, \ell - 1$),

define $D_\ell(X, a_1, \dots, a_{\ell-1}) \in \mathcal{B}_0(m, k) \otimes \mathcal{P}^i$ as the sum of pairs $b \otimes Y$ where $b \in \mathcal{B}_0(m, k)$ and Y is a generator of \mathcal{P}^i , satisfying the following conditions:

- the weights of b and Y satisfy

$$(5.10) \quad \mathbf{gr}(X) + \tau_i^{\mathbf{gr}}(\mathbf{gr}(a_1) + \dots + \mathbf{gr}(a_{\ell-1})) = \mathbf{gr}(b) + \mathbf{gr}(Y)$$

- There are generators X_0 and Y_0 with the same type (i.e. with the same label $\{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$) as X and Y respectively, so that $t(b) \otimes Y_0$ appears with non-zero multiplicity in $\delta_\ell^1(X_0, t(a_1), \dots, t(a_{\ell-1}))$.

Clearly, $D_\ell^1 = 0$ for $\ell > 3$. Define $\delta_1^1 = D_1$, we get

$$\delta_1^1(\mathbf{W}) = L_1 \otimes \mathbf{S}, \quad \delta_1^1(\mathbf{E}) = R_2 \otimes \mathbf{S}, \quad \delta_1^1(\mathbf{N}) = \delta_1^1(\mathbf{S}) = 0.$$

Lemma 5.8. Suppose that any $a_t \in \mathcal{I}(\mathbf{x}_t, \mathbf{x}_{t+1})$, then the projection of $D_\ell^1(X, a_1, \dots, a_{\ell-1})$ to $\mathcal{B}(m, k) \otimes \mathcal{P}^i$ vanishes; i.e. the maps D_ℓ induce well-defined maps

$$\delta_\ell^1: \mathcal{P}^i \otimes \overbrace{\mathcal{B}(m, k) \otimes \dots \otimes \mathcal{B}(m, k)}^{\ell-1} \rightarrow \mathcal{B}(m, k) \otimes \mathcal{P}^i,$$

for all $\ell = 2, 3$.

Proof. For the δ_1^1 operations, the result is obvious.

The operation δ_2^1 can be written as a sum of 16 terms, corresponding to the terms in the labels on Figure 5.4.

For example, consider $\delta_2^1(\mathbf{E}, a \cdot U_{i+1}^\ell)$, where $w_i(a) = w_{i+1}(a) = 0$ and $\ell > 0$. According to Equation 5.4, this can be written as a sum of two terms,

$$R_{i+1}R_i \cdot a \cdot U_i^{\ell-1} \otimes \mathbf{W} + a \cdot U_i^\ell \otimes \mathbf{E}.$$

We check that for any idempotent states \mathbf{x} and \mathbf{y} with $\mathbf{E} \cdot \mathbf{I}_\mathbf{x} \neq 0$ and $\mathbf{I}_\mathbf{x} \cdot a \cdot U_{i+1}^\ell \cdot \mathbf{I}_\mathbf{x} \neq 0$, if $\mathbf{I}_\mathbf{x} \cdot a \cdot U_{i+1}^\ell \in \mathcal{J}$, then the two output terms also vanish. If a is divisible by a monomial corresponding to a generating interval for (\mathbf{x}, \mathbf{y}) that is disjoint from $\{i, i+1\}$, each output elements are also divisible by such monomials, and the claim is clear. It remains to check the claim when $a \cdot U_{i+1}$ is the monomial corresponding to a generating interval $U_{i+1} \cdots U_\beta$ for the incoming element. In this case, the output is divisible by $U_{i+2} \cdots U_\beta$, which corresponds to a generating interval in both output algebras.

The other non-trivial checks include the following:

$$\begin{aligned} \delta_2^1(\mathbf{W}, U_\alpha \cdots U_i) &= (U_\alpha \cdots U_{i-1} U_{i+1}) \otimes \mathbf{W} + L_i L_{i+1} (U_\alpha \cdots U_{i-1}) \otimes \mathbf{E} \\ \delta_2^1(\mathbf{N}, U_{i+1} \cdots U_\beta \cdot L_i) &= (U_i \cdots U_\beta) \otimes \mathbf{W} + L_i L_{i+1} U_i (U_{i+2} \cdots U_\beta) \otimes \mathbf{E} \\ \delta_2^1(\mathbf{N}, U_\alpha \cdots U_{i-1} \cdot L_i) &= (U_\alpha \cdots U_i) \otimes \mathbf{W} + L_i L_{i+1} (U_\alpha \cdots U_{i-1}) \otimes \mathbf{E} \\ \delta_2^1(\mathbf{W}, U_\alpha \cdots U_i \cdot L_{i+1}) &= U_\alpha \cdots U_{i-1} \cdot L_i \cdot L_{i+1} \otimes \mathbf{N} \\ \delta_2^1(\mathbf{W}, U_\alpha \cdots U_{i-1} \cdot R_i) &= U_\alpha \cdots U_{i-1} \otimes \mathbf{N} \end{aligned}$$

where the monomials $U_\alpha \cdots U_i$ and $U_{i+1} \cdots U_\beta$ (with $\alpha \leq i$ and $\beta \geq i+2$) correspond to generating intervals for the input, for suitable choices of idempotents; it is then straightforward to check that the outputs are also zero. The remaining non-trivial checks are symmetric to the above ones (under the symmetry exchanging \mathbf{W} and \mathbf{E} , and L_i with R_{i+1}).

For δ_3^1 , we must show that if a_1 or $a_2 \in \mathcal{J}$, then the output $D_3^1(X, a_1, a_2)$ projects to zero in $\mathcal{B}(m, k) \otimes \mathcal{P}^i$. In cases where $\min(w_i(a_1) + w_i(a_2), w_{i+1}(a_1) + w_{i+1}(a_2)) = \frac{1}{2}$, i.e. when $(t(a_1), t(a_2)) \in \{(R_1, R_2 U_2^t), (L_2, L_1 U_1^n), (L_2, U_1^{n+1}), (R_1, U_2^{t+1})\}$, are considered separately. In all these cases, the generating intervals for a_1 and a_2 do not contain i or $i+1$, so the result is obvious.

For the remaining cases, we have the following stronger assertion: if a_1 and a_2 satisfy

$$(5.11) \quad \min(w_i(a_1) + w_i(a_2), w_{i+1}(a_1) + w_{i+1}(a_2)) > \frac{1}{2},$$

and $a_1 \cdot a_2 \in \mathcal{J}$ (i.e. $a_1 \cdot a_2$ projects to zero in $\mathcal{B}(m, k)$), and with $\mathbf{I}_\mathbf{x} \cdot a_1 = a_1$ then $D_3^1(X, a_1, a_2) \in \mathcal{J}$. For instance, the terms in $\delta_3^1(\mathbf{S}, a_1, a_2)$ in $\mathcal{B}(2)$ that output $L_2 U_1^t U_2^n \otimes \mathbf{E}$ (all of which satisfy Equation (5.11)) give rise to actions with $a_1, a_2 \in \mathcal{B}(k, m)$ with $\mathbf{I}_\mathbf{x} \cdot a_1 \cdot a_2 = a_1 \cdot a_2$ with

$$i \notin \mathbf{p}, \quad w_i(a_1 \cdot a_2) = n + 1 \geq 1, \quad w_{i+1}(a_1 \cdot a_2) = t \geq 1$$

and output algebra element $b = \mathbf{I}_\mathbf{x} \cdot b \cdot \mathbf{I}_\mathbf{y}$ with

$$i \in \mathbf{y}, \quad i+1 \notin \mathbf{y} \quad w_i(b) = t \geq 1$$

It is easy to see that if $a_1 \cdot a_2$ is divisible by a monomial corresponding to a generating interval of the form $U_\alpha \cdots U_i$, then so is b ; checking cases where $a_1 \cdot a_2$ is divisible by other generating intervals is even more straightforward. Remaining cases where the output contains \mathbf{W} work similarly. Cases where the output contains

\mathbf{N} are easily verified, as well. The stronger statement (following Equation (5.11)) now follows. \square

For example,

$$\delta_2^1(\mathbf{N}, R_{i+2}R_{i+1}) = R_{i+2}R_{i+1}R_i \otimes \mathbf{W} + R_{i+2}U_i \otimes \mathbf{E}.$$

As another example, choose R_j with $j \neq i$ or $i+1$. Then, for all generators X , $\delta_2^1(X, R_j) = R_j \otimes X$. (Note that in this example, $t(R_j) = 1$.)

As another example,

$$\delta_3^1(\mathbf{S}, R_i, R_{i+2}R_{i+1}) = \delta_3^1(\mathbf{S}, R_iR_{i+2}, R_{i+1}) = R_iR_{i+2} \otimes \mathbf{E}.$$

Let $\delta_\ell^1 = 0$ for all $\ell \geq 4$.

It will be useful to have the following characterization of δ_3^1 :

Lemma 5.9. *The operation $\delta_3^1(X, a_1, a_2)$ contains only those terms $b \otimes Y$ where both Equation (5.10) holds, and one of the following two conditions holds:*

- (C-1) $(t(a_1), t(a_2)) \in \{(R_1, R_2U_2^t), (L_2, L_1U_1^n), (L_2, U_1^{n+1}), (R_1, U_2^{t+1})\}$ and $t(b) \otimes t(Y)$ appears with non-zero multiplicity in $\delta_3^1(t(X), t(a_1), t(a_2))$;
- (C-2) $a_1 \cdot a_2 \neq 0$, $I(a_2, Y) \neq 0$ and $I(t(a_1), I(t(a_2), t(Y))) \neq I(t(a_1) \cdot t(a_2), t(Y))$.

Proof. In the proof of Lemma 5.8, we showed that if we are not in Case (C-1), then Equation (5.11) holds, and so we conclude that $a_1 \cdot a_2 \neq 0$. Thus, our description of δ_3^1 follows from Lemma 5.5. \square

Proposition 5.10. *The above maps give \mathcal{P}^i the structure of a type DA bimodule over $\mathcal{B}(m, k)$ - $\mathcal{B}(m, k)$.*

Proof. The proof of Proposition 5.2 adapts, with a few remarks. In the verification of the \mathcal{A}_∞ relation with two algebra inputs, that proof decomposed according to whether we were working in $\mathcal{B}(2, 1)$ or $\mathcal{B}(2, 2)$. In the present case, when $(t(a_1), t(a_2)) \in \{(R_1, R_2U_2^t), (L_2, L_1U_1^n), (L_2, U_1^{n+1}), (R_1, U_2^{t+1})\}$, the \mathcal{A}_∞ relation holds exactly as it did in $\mathcal{B}(2, 1)$. Otherwise, if $a_1 \cdot a_2 \neq 0$, the proof of the \mathcal{A}_∞ relation for $\mathcal{B}(2, 2)$ applies, using Lemma 5.9. Finally, if $a_1 \cdot a_2 = 0$, then by Lemma 5.9, the term involving δ_3^1 (F or G) vanishes. The term D vanishes by Lemma 5.8. The verification of the \mathcal{A}_∞ relation in $\mathcal{B}(2, 2)$ shows that the remaining possible non-zero term, which is of type E , has the same contribution as a term of type F , G , or D , all of which contribute 0.

Consider next the case of three algebra inputs a_1 , a_2 , and a_3 . When $a_1 \cdot a_2 \cdot a_3 = 0$, the verification (now in the proof of Lemma 5.6) works as it did in $\mathcal{B}(2, 1)$. In the remaining cases, the earlier proof of the \mathcal{A}_∞ relation in $\mathcal{B}(2, 2)$ (contained in the proof of Lemma 5.6) still applies, as it hinges on the description of δ_3^1 (Lemma 5.5) which still holds in this case (according to Lemma 5.9). \square

5.3. Adding C_i , locally. We extend the local bimodule P defined over $\mathcal{B}(2)$ to a bimodule ${}^{\mathcal{B}(2, \tau(\mathcal{S}))}P_{\mathcal{B}(2, \mathcal{S})}$, where \mathcal{S} is a non-empty subset of $\{1, 2\}$.

In the local module P , there were δ_2^1 actions connecting generators of types $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ to $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ or \mathbf{S} to \mathbf{S} . We extend these to similar actions so that

$$(5.12) \quad \delta_2^1(X, C_1 \cdot a) = C_2 \cdot \delta_2^1(X, a) \quad \delta_2^1(X, C_2 \cdot a) = C_1 \cdot \delta_2^1(X, a)$$

Similarly, we extend the previous δ_3^1 actions so that

$$\begin{aligned} C_2 \cdot \delta_2^1(X, a_1, a_2) &= \delta_3^1(X, C_1 \cdot a_1, a_2) = \delta_3^1(X, a_1, C_1 \cdot a_2) \\ C_1 \cdot \delta_2^1(X, a_1, a_2) &= \delta_3^1(X, C_2 \cdot a_1, a_2) = \delta_3^1(X, a_1, C_2 \cdot a_2) \end{aligned}$$

For example, $\delta_3^1(C_1 R_1, C_2 R_2) = \delta_3^1(C_1 C_2 R_1, R_2) = R_1 C_1 C_2 \otimes \mathbf{E}$. We specify further δ_2^1 actions from \mathbf{S} to $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$:

$$\begin{aligned} \delta_2^1(\mathbf{S}, C_2) &= R_1 \otimes \mathbf{W} & \delta_2^1(\mathbf{S}, C_1 C_2) &= C_2 R_1 \otimes \mathbf{W} + C_1 L_2 \otimes \mathbf{E} \\ \delta_2^1(\mathbf{S}, C_1) &= L_2 \otimes \mathbf{E} & \delta_2^1(\mathbf{S}, U_1 C_1 C_2) &= U_1 C_2 L_2 \otimes \mathbf{E} \\ \delta_2^1(\mathbf{S}, U_1 C_2) &= U_1 L_2 \otimes \mathbf{E} & \delta_2^1(\mathbf{S}, C_1 C_2 U_2) &= C_1 R_1 U_2 \otimes \mathbf{W} \\ \delta_2^1(\mathbf{S}, C_1 U_2) &= R_1 U_2 \otimes \mathbf{W} & \delta_2^1(\mathbf{S}, R_1 C_1 C_2) &= R_1 C_2 \otimes \mathbf{N} \\ \delta_2^1(\mathbf{S}, R_1 C_2) &= R_1 \otimes \mathbf{N} & \delta_2^1(\mathbf{S}, C_1 L_2 C_2) &= C_1 L_2 \otimes \mathbf{N} \\ \delta_2^1(\mathbf{S}, L_2 C_1) &= L_2 \otimes \mathbf{N} & \delta_2^1(\mathbf{S}, R_1 C_1 U_2 C_2) &= R_1 C_1 U_2 \otimes \mathbf{N} \\ \delta_2^1(\mathbf{S}, R_1 C_1 U_2) &= R_1 U_2 \otimes \mathbf{N} & \delta_2^1(\mathbf{S}, U_1 C_1 L_2 C_2) &= U_1 L_2 C_2 \otimes \mathbf{N} \\ \delta_2^1(\mathbf{S}, U_1 L_2 C_2) &= L_2 U_1 \otimes \mathbf{N} \end{aligned}$$

These are extended to commute with multiplication by $U_1 U_2$ (as in Equation (5.5)).

By Equation (5.12), the actions $\delta_2^1(\mathbf{S}, 1) = \mathbf{S}$ gives rise to actions $\delta_2^1(\mathbf{S}, C_2) = C_1 \otimes \mathbf{S}$. The first action listed above, i.e. the relation $\delta_2^1(\mathbf{S}, C_2) = R_1 \otimes \mathbf{W}$, is now a consequence of this fact, together with the \mathcal{A}_∞ relation with module element \mathbf{S} , the single algebra input C_2 , and output module element \mathbf{S} . The second action on the above list follows symmetrically. The actions on the left column of rows three, five, and seven respectively are forced by the following δ_3^1 actions (with $\mathcal{S} = \emptyset$)

$$\begin{aligned} \delta_3^1(\mathbf{S}, U_1, U_2) &= L_2 U_1 \otimes \mathbf{E} \\ \delta_3^1(\mathbf{S}, L_2, U_1) &= L_2 \otimes \mathbf{N} \\ \delta_3^1(\mathbf{S}, R_1 U_2, U_1) &= R_1 U_2 \otimes \mathbf{N} \end{aligned}$$

and the \mathcal{A}_∞ relations with inputs (\mathbf{S}, U_1, C_2) , (\mathbf{S}, L_2, C_1) and $(\mathbf{S}, R_1 U_2, C_1)$ respectively. The remaining actions on the left column follow symmetrically. The actions in the second column follow from actions from the first column of the form (\mathbf{S}, da) and \mathcal{A}_∞ relations with inputs (\mathbf{S}, a) .

Lemma 5.11. *For any $\mathcal{S} \subset \{1, 2\}$, the above actions induce a type DA bimodule structure on ${}^{\mathcal{B}(2, \tau(\mathcal{S}))}P_{\mathcal{B}(2, \mathcal{S})}$.*

Proof. It suffices to prove the lemma in the case where $\mathcal{S} = \{1, 2\}$. For example, if $\mathcal{S} = \{1\}$, the outputs of the δ_j^1 actions lie in the subalgebra $\mathcal{B}(2, \{2\})$.

We consider the \mathcal{A}_∞ relation with n incoming algebra elements a_1, \dots, a_n .

It is a straightforward verification to check that the actions defined above are consistent with the \mathcal{A}_∞ relation with $n = 1$ input.

When $n = 2$, and the incoming generator is not \mathbf{S} , the \mathcal{A}_∞ relation follows from Equation (5.12) together with the \mathcal{A}_∞ relation with $\mathcal{S} = \emptyset$.

When $n = 2$ and the incoming generator is \mathbf{S} , label the terms in the \mathcal{A}_∞ relation as in Equation (5.9). The key point to verifying this relation now is that when differentiating a_1 or a_2 (in terms B or C above), the U_1 or U_2 -power can change by at most one so the corresponding δ_3^1 action with a_1 and a_2 is non-zero when the action with (da_1, a_2) and (a_1, da_2) is. However, there are borderline cases where this change in the U_1 or the U_2 power is enough to turn off one of those actions. These are precisely the cases either the product $a_1 a_2$ acts non-trivially (i.e. where D contributes), or there is an iterated δ_2^1 (i.e. E contributes).

For example, consider the case where $a_1 = U_1^n C_1$, and $a_2 = U_2^t$ with $n, t \geq 0$, and consider the terms where the output generator has type \mathbf{E} . We have the following non-zero terms in the \mathcal{A}_∞ relation:

$$A \neq 0 \Leftrightarrow 0 \leq n-1 < t, \quad B \neq 0 \Leftrightarrow 0 \leq n < t, \quad D \neq 0 \Leftrightarrow n = t \geq 0, \quad E \neq 0 \Leftrightarrow n = 0, t \geq 0;$$

Thus, for all choices of $n, t \geq 0$, there are either no non-zero terms or exactly two, and so the \mathcal{A}_∞ relation holds. Other cases where $n = 2$ and the initial generator is of type \mathbf{S} work similarly.

Consider the case where $n = 3$. Recall that the algebra has a filtration in $\{0, 1\} \times \{0, 1\}$ given by the functions $(\mathbf{m}_1, \mathbf{m}_2)$, with the property that if a is a pure algebra element not divisible by C_j , then $\mathbf{m}_j(a) = 0$ and $\mathbf{m}_j(C_j a) = 1$. Clearly, the operations δ_ℓ^1 respect this filtration, in the sense that if a $(\mathbf{m}_1, \mathbf{m}_2)$ -homogeneous element $b \otimes Y$ appears with non-zero multiplicity in $\delta_\ell^1(a_1, \dots, a_{\ell-1})$, and each a_i is homogeneous (with respect to μ), then for $j = 1, 2$,

$$(\mathbf{m}_1(b), \mathbf{m}_2(b)) \leq \sum_{k=1}^{\ell-1} (\mathbf{m}_1(a_k), \mathbf{m}_2(a_k)).$$

For the terms that preserve this $\{0, 1\} \times \{0, 1\}$ -filtration, the \mathcal{A}_∞ relation with 3 inputs is an immediate consequence of Proposition 5.10. Consider next terms that drop filtration level by one. Note that each δ_ℓ^1 action of this type has $\ell = 2$ and end in the span of $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$, while all δ_3^1 actions start from \mathbf{S} . It follows that there are no such terms that appear in the \mathcal{A}_∞ relation with 3 inputs. (It also follows that there are no terms of this kind in the \mathcal{A}_∞ relation which drop by more than 1.)

For $n > 3$, the \mathcal{A}_∞ relation is easy. \square

5.4. The general case of \mathcal{P}^i . Let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S})$, where $0 \leq k \leq m+1$ and $\mathcal{S} \subset \{1, \dots, m\}$ is arbitrary; and let $\mathcal{B}_2 = \mathcal{B}(m, k, \tau(\mathcal{S}))$. In cases where $\mathcal{S} \cap \{i, i+1\} \neq \emptyset$, we must modify our earlier constructions as follows.

Extend the type $t(a)$ now to be a monomial in $U_1, U_2, L_1, L_2, C_1, C_2, R_1, R_2$, so that

$$\begin{aligned} t(C_i \cdot a) &= C_1 \cdot t(a) & \text{if } C_i \cdot a \neq 0 \\ t(C_{i+1} \cdot a) &= C_2 \cdot t(a) & \text{if } C_{i+1} \cdot a \neq 0; \end{aligned}$$

and $t(a)$ is as defined before when a is a pure algebra element not divisible by C_i or C_{i+1} .

Let $t(\mathcal{S}) \subset \{1, 2\}$ be the set with $1 \in t(\mathcal{S})$ iff $i \in \mathcal{S}$ and $2 \in t(\mathcal{S})$ iff $i+1 \in \mathcal{S}$.

Definition 5.12. For $X \in \mathcal{P}^i$ and a sequence of pure algebra elements $a_1, \dots, a_{\ell-1}$ in $\mathcal{B}_0(m, k, \mathcal{S})$, so that there exist a sequence of idempotent states $\mathbf{x}_0, \dots, \mathbf{x}_\ell$ with

- $X = \mathbf{I}_{\mathbf{x}_0} \cdot X \cdot \mathbf{I}_{\mathbf{x}_1}$
- $a_t = \mathbf{I}_{\mathbf{x}_t} \cdot a_t \cdot \mathbf{I}_{\mathbf{x}_{t+1}}$ for $t = 1, \dots, \ell$
- \mathbf{x}_t and \mathbf{x}_{t+1} are close enough (for $k = 0, \dots, \ell - 1$),

define $D_\ell(X, a_1, \dots, a_{\ell-1}) \in \mathcal{B}_0(m, k, \tau(\mathcal{S})) \otimes \mathcal{P}^i$ as the sum of pairs $b \otimes Y$ where $b \in \mathcal{B}_0(m, k, t(\tau(\mathcal{S})))$ and Y is a generator of \mathcal{P}^i , satisfying the conditions of Definition 5.7, with the understanding that now $\delta_\ell^1(X_0, t(a_1), \dots, t(a_{\ell-1}))$ is computed using the crossing over ${}^{\mathcal{B}(2, t(\mathcal{S}))}P_{\mathcal{B}(2, t(\mathcal{S}))}$; and the additional condition that for any $k \neq i$ or $i + 1$, and any $m = 1, \dots, \ell - 1$,

$$D_\ell(X, a_1, \dots, C_k \cdot a_m, \dots, a_{\ell-1}) = C_k \cdot D_\ell(X, a_1, \dots, a_m, \dots, a_{\ell-1}).$$

Lemma 5.13. *If any $a_t \in \mathcal{I}(\mathbf{x}_t, \mathbf{x}_{t+1})$, then the projection of $D_\ell^1(X, a_1, \dots, a_{\ell-1})$ to $\mathcal{B}(m, k) \otimes \mathcal{P}^i$ vanishes; i.e. the maps D_ℓ induce well-defined maps*

$$\delta_\ell^1: \mathcal{P}^i \otimes \overbrace{\mathcal{B}(m, k, \mathcal{S}) \otimes \dots \otimes \mathcal{B}(m, k, \mathcal{S})}^{\ell-1} \rightarrow \mathcal{B}(m, k, \tau(\mathcal{S})) \otimes \mathcal{P}^i,$$

for all $\ell = 2, 3$.

Proof. Lemma 5.8 takes care of most of this; we must check further the additional δ_2^1 actions from \mathbf{S} to $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ listed in the beginning of this subsection; but this is straightforward. \square

Proposition 5.14. *The above maps give \mathcal{P}^i the structure of a type DA bimodule over $\mathcal{B}(m, k, \tau(\mathcal{S}))$ - $\mathcal{B}(m, k, \mathcal{S})$.*

Proof. This follows easily from Proposition 5.10, handling terms with C_i inputs as in Lemma 5.11. \square

5.5. The negative crossing. Consider the map $\mathcal{R}: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \mathcal{S})$ from Section 3.6. Recall that $o(a \cdot b) = o(b) \cdot o(a)$, $o(\mathbf{I}_{\mathbf{x}}) = \mathbf{I}_{\mathbf{x}}$, $o(L_t) = R_t$, $o(R_t) = L_t$, $o(U_t) = U_t$ and $o(C_j) = C_j$ for all $t = 1, \dots, m$ and $j \in \mathcal{S}$.

Let \mathcal{N}^i be generated by the same generators $\{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$ as before. If $\delta_1^1(X) = b \otimes Y$ in \mathcal{P}^i , then $\delta_1^1(Y) = o(b) \otimes X$ in \mathcal{N}^i . If $\delta_2^1(X, a) = o(b) \otimes Y$ in \mathcal{P}^i , then $\delta_2^1(Y, o(a)) = o(b) \otimes X$ in \mathcal{N}^i . If $\delta_3^1(X, a_1, a_2) = b \otimes Y$ in \mathcal{P}^i , then $\delta_3^1(Y, o(a_2), o(a_1)) = o(b) \otimes X$.

More succinctly, the opposite module of \mathcal{P}^i is identified with

$${}_{\mathcal{B}_2}(\overline{\mathcal{P}}^i)^{\mathcal{B}_1} \cong {}^{\mathcal{B}_1^{\text{op}}} \overline{\mathcal{P}}_{\mathcal{B}_2^{\text{op}}}^i = {}^{\mathcal{B}_1} \mathcal{N}_{\mathcal{B}_2}^i;$$

under the identification of $\mathcal{B}_t^{\text{op}} \cong \mathcal{B}_t$ for $t = 1, 2$ (Equation (3.9)).

For example, in \mathcal{N}^i , we have actions

$$\begin{aligned} \delta_1^1(\mathbf{S}) &= R_i \otimes \mathbf{W} + L_{i+1} \otimes \mathbf{E} \\ \delta_2^1(\mathbf{E}, R_i) &= R_{i+1} R_i \otimes \mathbf{N} \\ \delta_3^1(\mathbf{W}, R_i U_i^n, R_{i+1}) &= R_{i+1} U_{i+1}^n \otimes \mathbf{S}. \end{aligned}$$

Proposition 5.15. *The above maps give \mathcal{N}^i the structure of a type DA bimodule over $\mathcal{B}(m, k, \tau(\mathcal{S}))$ - $\mathcal{B}(m, k, \mathcal{S})$.*

Proof. This is a formal consequence of the above definition. \square

5.6. Gradings and partial Kauffman states. As noted earlier, \mathcal{P}^i is adapted to the underlying manifold. Thus, its grading set takes values in $H^1(W, \partial W)$. We can specialize to a simply \mathbb{Q} -graded-graded setting as follows.

Recall (Equation (3.6)) that the algebra has an Alexander grading with values in \mathbb{Q} , obtained from the map $\phi: \frac{1}{2}\mathbb{Z}^m \rightarrow \frac{1}{2}\mathbb{Z}$ defined by

$$\phi(e_i) = \begin{cases} -1 & \text{if } i \in \mathcal{S} \\ 1 & \text{if } i \notin \mathcal{S}. \end{cases}$$

There is an induced grading on the bimodule, given by

$$(5.13) \quad A(X) = \phi(\mathbf{gr}(X));$$

which we can think of, more abstractly, as the evaluation of the grading, thought of as an element of $H^1(W, \partial W)$, against the element $[W, \partial] \in H_1(W, \partial W)$ induced by the orientation of W .)

It is now an immediate consequence of Equation (5.1) that

$$A(X, a_1, \dots, a_\ell) = A(X) + A(a_1) + \dots + A(a_\ell) = A(b) + A(Y),$$

if $b \otimes Y$ appears with non-zero multiplicity in $\delta_{\ell+1}^1(X, a_1, \dots, a_\ell)$.

Proposition 5.16. *The \mathbb{Q} -valued Alexander grading on generators from Equation (5.13) for the crossing bimodules is computed by the local Kauffman contributions displayed in Figure 2.*

Proof. This is a straightforward check using Equation (5.13) Equation (4.5), and (4.7), considering all the possible orientations of the braidlike positive and negative crossing. \square

6. BRAID RELATIONS

We prove that the DA bimodules \mathcal{P}^i and \mathcal{N}^i satisfy the following braid relations:

Theorem 6.1. *Fix m, k with $0 \leq k \leq m+1$, i with $1 \leq i \leq m-1$, and $\mathcal{S} \subset \{1, \dots, m\}$. Let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S})$ and $\mathcal{B}_2 = \mathcal{B}(m, k, \tau_i(\mathcal{S}))$. Then,*

$$(6.1) \quad {}^{\mathcal{B}_1}\mathcal{P}_{\mathcal{B}_2}^i \boxtimes {}^{\mathcal{B}_2}\mathcal{N}_{\mathcal{B}_1}^i \simeq {}^{\mathcal{B}_1}\text{Id}_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_1}\mathcal{N}_{\mathcal{B}_2}^i \boxtimes {}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i$$

Given $j \neq i$ with $1 \leq j \leq m-1$, let

$$\mathcal{B}_3 = \mathcal{B}(m, k, \tau_j \tau_i(\mathcal{S})) \quad \text{and} \quad \mathcal{B}_4 = \mathcal{B}(m, k, \tau_j(\mathcal{S})).$$

If $|i-j| > 1$

$$(6.2) \quad {}^{\mathcal{B}_3}\mathcal{P}_{\mathcal{B}_2}^j \boxtimes {}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i \simeq {}^{\mathcal{B}_3}\mathcal{P}_{\mathcal{B}_4}^i \boxtimes {}^{\mathcal{B}_4}\mathcal{P}_{\mathcal{B}_1}^j.$$

while if $j = i+1$, let

$$\mathcal{B}_5 = \mathcal{B}(m, k, \tau_i \tau_{i+1} \tau_i(\mathcal{S})) \quad \text{and} \quad \mathcal{B}_6 = \mathcal{B}(m, k, \tau_i \tau_{i+1}(\mathcal{S}));$$

then,

$$(6.3) \quad {}^{\mathcal{B}_5}\mathcal{P}_{\mathcal{B}_3}^i \boxtimes {}^{\mathcal{B}_3}\mathcal{P}_{\mathcal{B}_2}^{i+1} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i \simeq {}^{\mathcal{B}_5}\mathcal{P}_{\mathcal{B}_6}^{i+1} \boxtimes {}^{\mathcal{B}_6}\mathcal{P}_{\mathcal{B}_4}^i \boxtimes {}^{\mathcal{B}_4}\mathcal{P}_{\mathcal{B}_1}^{i+1}.$$

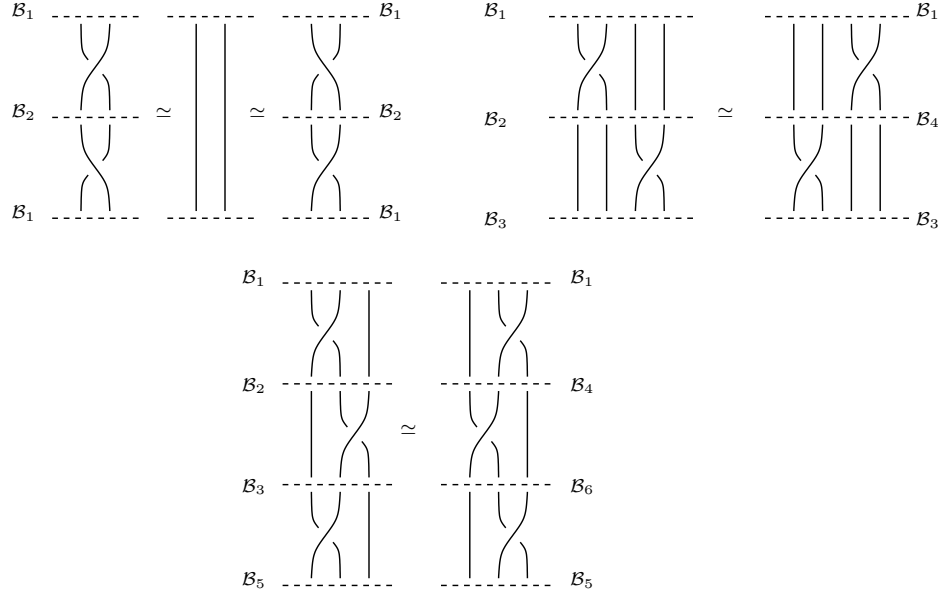


FIGURE 21. Bimodules and algebras appearing in Theorem 6.1.

We will prove the above with the help of the following:

Lemma 6.2. *Fix m, k with $0 \leq k \leq m+1$, i with $1 \leq i \leq m-1$, and $\mathcal{S} \subset \{1, \dots, m\}$. Let*

$$\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}) \quad \mathcal{B}_2 = \mathcal{B}(m, k, \tau_i(\mathcal{S})), \quad \mathcal{B}'_1 = \mathcal{B}(m, m+1-k, \{1, \dots, m\} \setminus \mathcal{S}).$$

Let ${}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i$ be the DA bimodule from Section 5, and let ${}^{\mathcal{B}_1, \mathcal{B}'_1}\mathcal{K}$ be the canonical type DD bimodule from Section 3.7. Then,

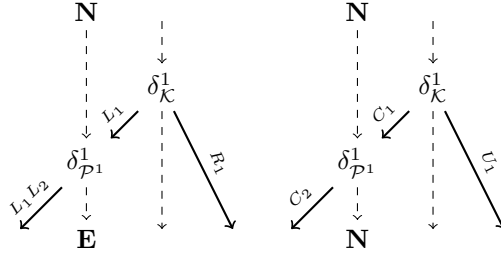
$$(6.4) \quad {}^{\mathcal{B}_2}\mathcal{P}_{\mathcal{B}_1}^i \boxtimes^{\mathcal{B}_1, \mathcal{B}'_1} \mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}'_1}\mathcal{P}_i$$

$$(6.5) \quad {}^{\mathcal{B}_2}\mathcal{N}_{\mathcal{B}_1}^i \boxtimes^{\mathcal{B}_1, \mathcal{B}'_1} \mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}'_1}\mathcal{N}_i$$

where the type DD bimodule appearing on the right is the one defined in Section 7.

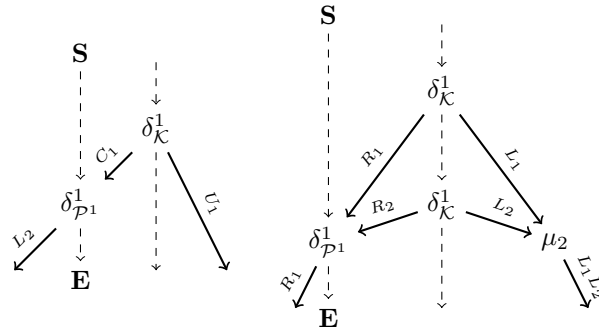
Proof. We start by verifying Equation (6.4); and for simplicity, we assume $i = 1$. The computation depends on $\mathcal{S} \cap \{1, 2\}$.

We start with the case where $\{1, 2\} \subset \mathcal{S}$. In fact, in this case, $\mathcal{P}^i \boxtimes \mathcal{K} = \mathcal{P}_i$. The arrows connecting **N**, **W**, and **E** are all induced from δ_2 actions on \mathcal{P}^i . For example, the differential from **N** to **E** labelled by $L_1 L_2 \otimes R_1$ from Figure 4.2 is obtained by pairing the term $L_1 \otimes R_1$ term in the canonical DD bimodule pairs with the action $\delta_2^1(\mathbf{N}, L_1) = L_1 L_2 \otimes \mathbf{E}$ in the \mathcal{P}^1 , as shown on the left diagram:



On the right, we demonstrate how to construct the term $(C_2 \otimes U_1) \otimes \mathbf{N}$ in $\delta^1(\mathbf{N})$ (of Type (P-2)).

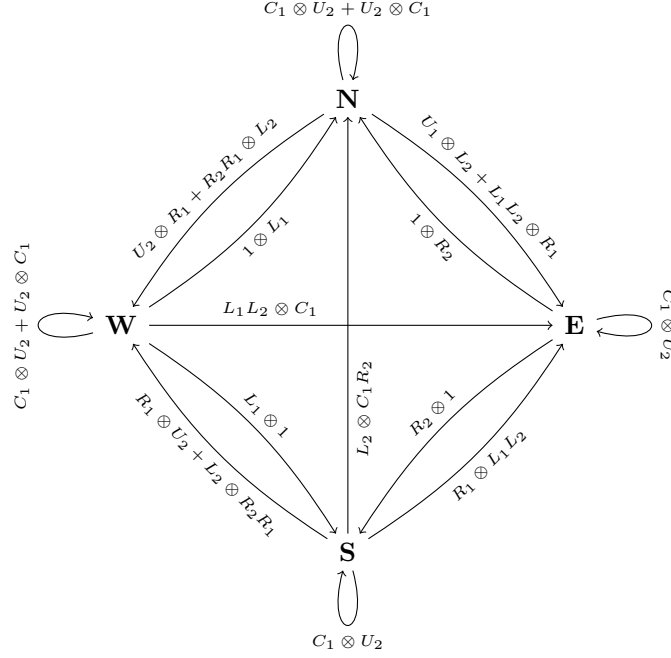
The differentials into **S** come from the δ_1^1 action on \mathcal{P}^i , and the differentials out of **S** come from δ_2^1 and δ_3^1 actions on \mathcal{P}^i . For example, the two terms out of **S** involving the generator **E** arise as follows:



(Note that since \mathcal{K} bimodule is a left/left bimodule, the multiplication appearing on the right is taking place in $(\mathcal{B}'_1)^{\text{op}}$.)

Terms of Type (P-1) are easily constructed from pairing differentials in the identity DD bimodule with the part of \mathcal{P}^i that behaves like an identity bimodule.

Consider next the case where $\mathcal{S} \cap \{1, 2\} = \{2\}$. Then, $\mathcal{P}^1 \boxtimes \mathcal{K}$ is given by:



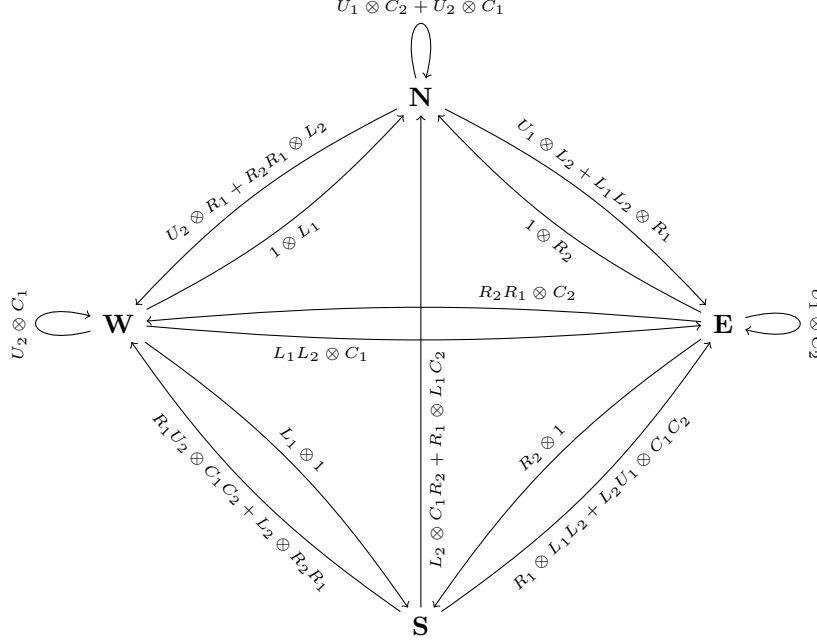
and outside actions (i.e. of Type (P-1) and (P-2) with $j \neq 1, 2$). Consider the map $h^1: \mathcal{P}^1 \boxtimes \mathcal{K} \rightarrow \mathcal{P}_1$

$$h^1(X) = \begin{cases} \mathbf{S} + (L_2 \otimes C_1) \otimes \mathbf{E} & \text{if } X = \mathbf{S} \\ X & \text{otherwise.} \end{cases}$$

Let $g^1: \mathcal{P}_1 \rightarrow \mathcal{P}^1 \boxtimes \mathcal{K}$ be given by the same formula. It is easy to verify that h^1 and g^1 are homomorphisms of type DD structures, $h^1 \circ g^1 = \text{Id}$, and $g^1 \circ h^1 = \text{Id}$.

The case where $\mathcal{S} \cap \{1, 2\} = \{1\}$ works similarly.

When $\mathcal{S} \cap \{1, 2\} = \emptyset$, $\mathcal{P}^1 \boxtimes \mathcal{K}$ is given by



Consider the map $h^1: \mathcal{P}^1 \boxtimes \mathcal{K} \rightarrow \mathcal{P}_1$

$$h^1(X) = \begin{cases} \mathbf{S} + (L_2 \otimes C_1) \cdot \mathbf{E} + (R_1 \otimes C_2) \cdot \mathbf{W} & \text{if } X = \mathbf{S} \\ X & \text{otherwise.} \end{cases}$$

Let $g^1: \mathcal{P}_1 \rightarrow \mathcal{P}^1 \boxtimes \mathcal{K}$ be given by the same formula. It is easy to verify that h^1 and g^1 are homomorphisms of type DD structures, $h^1 \circ g^1 = \text{Id}$, and $g^1 \circ h^1 = \text{Id}$.

Equation (6.4) in cases where $i \neq 1$ works the same way. Equation (6.5) can be proved similarly. (Or alternatively, it can be seen as a consequence of Equation (6.4) and the symmetry of the bimodules, phrased in terms of the opposite algebras.) \square

Next we verify Equation (6.1).

Lemma 6.3. *Equation (6.1) holds.*

Proof. Since $(\mathcal{P}^i \boxtimes \mathcal{N}^i) \boxtimes \mathcal{K} \simeq \mathcal{P}^i \boxtimes (\mathcal{N}^i \boxtimes \mathcal{K}) \simeq \mathcal{P}^i \boxtimes \mathcal{N}_i$ (by associativity of \boxtimes and Lemma 6.2), the verification that $\mathcal{P}^i \boxtimes \mathcal{N}^i \simeq \text{Id}_{\mathcal{B}_1}$, will follow from the identity

$$(6.6) \quad \mathcal{P}^i \boxtimes \mathcal{N}_i \simeq \mathcal{K},$$

which we verify presently. For notational simplicity, we assume that $i = 1$.

The verification of Equation (6.6) can be divided into cases, according to $\mathcal{S} \cap \{1, 2\}$. Consider first the case where $\mathcal{S} \cap \{1, 2\} = \{1, 2\}$. Classify the generators of $\mathcal{P}^1 \boxtimes \mathcal{N}_1$ into types, labeled XY , where $X, Y \in \{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$, where the first symbol X denotes the generator type in \mathcal{P}^1 and Y denotes the generator type in \mathcal{N}_1 , as illustrated in Figure 22.

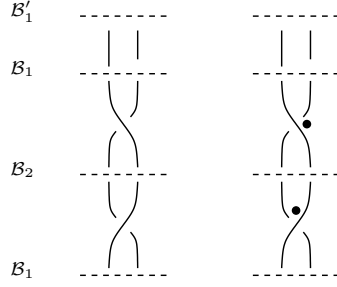
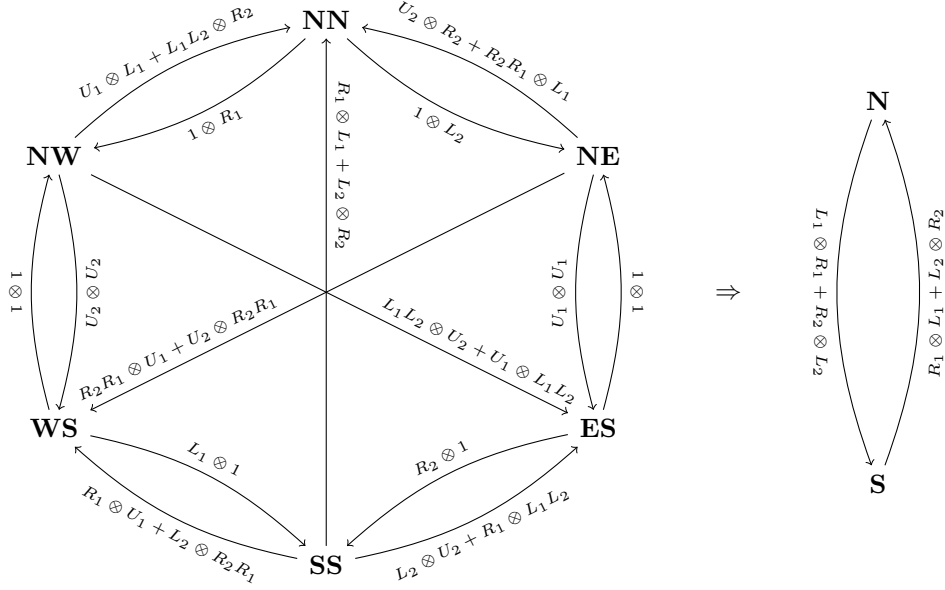


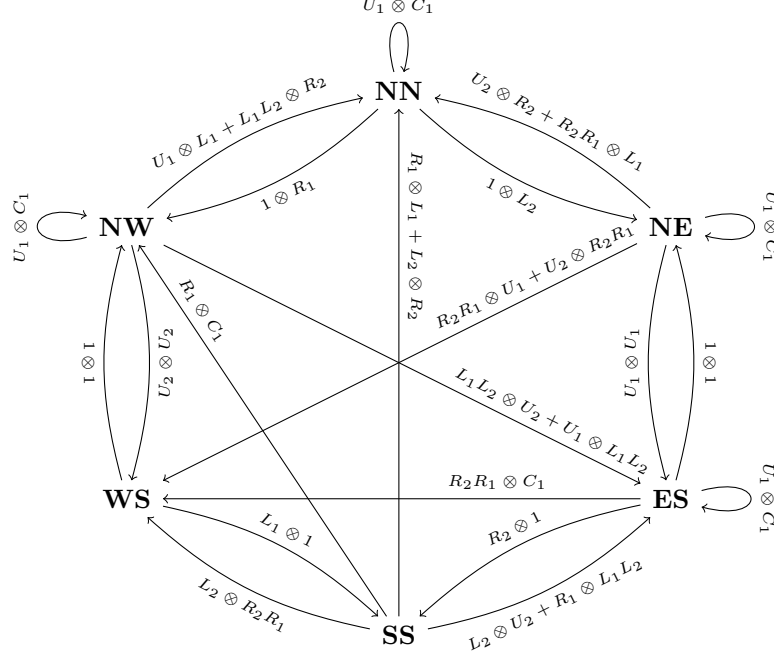
FIGURE 22. Tensoring bimodules on the left; at the right, the generator type of $\mathbf{NE} = \mathbf{N} \boxtimes \mathbf{E} \boxtimes \mathbf{1}$.

By a straightforward computation, we find the following kinds of differential: the outside actions, $C_1 \otimes U_1$, $C_2 \otimes U_2$, and additional arrows indicated on the diagram on the left:



Canceling arrows (i.e. setting $\mathbf{N} = \mathbf{NN} + (1 \otimes R_1)\mathbf{WS} + (1 \otimes L_2)\mathbf{ES}$ and $\mathbf{S} = \mathbf{SS}$; and noting that the quotient complex is acyclic) gives the diagram on the right (in addition to the $C_i \otimes U_i$ arrows for $i = 1, 2$ and the outside arrows). This is, in fact, the canonical DD bimodule, verifying Equation (6.6).

Similarly, if $\mathcal{S} \cap \{1, 2\} = \{2\}$, we compute the bimodule to be



We have suppressed the outside arrows, and additional arrows of the form $C_2 \otimes U_2$ that connect every generator to themselves. By contrast, we did include self-arrows of the form $U_1 \otimes C_1$, since not all generator types have these kinds of terms. We can reduce to the previous case by changing basis; replacing **SS** by $\mathbf{SS} + (R_1 \otimes C_1) \otimes \mathbf{WS}$. A similar computation works for $\mathcal{S} \cap \{1, 2\} = \{1\}$, only now the reduction to the previous case involves the change of basis $\mathbf{SS} + (L_2 \otimes C_2) \otimes \mathbf{ES}$. Finally, when $\mathcal{S} \cap \{1, 2\} = \emptyset$, we get a bimodule which can be reduced to the earlier case by the basis change $\mathbf{SS} + (R_1 \otimes C_1) \otimes \mathbf{WS} + (L_2 \otimes C_2) \otimes \mathbf{ES}$.

Once again, cancelling arrows we find that this gives the identity DD bimodule. The other two computations needed to verify Equation (6.6) work similarly.

Having verified that $\mathcal{P}^i \boxtimes \mathcal{N}^i \simeq {}^{\mathcal{B}_1} \text{Id}_{\mathcal{B}_1}$, the verification that $\mathcal{N}^i \boxtimes \mathcal{P}^i \simeq {}^{\mathcal{B}_1} \text{Id}_{\mathcal{B}_1}$ is a formality:

$$\begin{aligned} \text{Id} &= \overline{\text{Id}} \simeq \overline{{}^{\mathcal{B}_1} \mathcal{P}_{\mathcal{B}_2}^i \boxtimes {}^{\mathcal{B}_2} \mathcal{N}_{\mathcal{B}_1}^i} \simeq {}_{\mathcal{B}_1} \overline{\mathcal{N}}^i {}^{\mathcal{B}_2} \boxtimes {}_{\mathcal{B}_2} \overline{\mathcal{P}}^i {}^{\mathcal{B}_1} \\ &= {}^{\mathcal{B}_1^{\text{op}}} \overline{\mathcal{P}}^i {}_{\mathcal{B}_2^{\text{op}}} \boxtimes {}^{\mathcal{B}_2^{\text{op}}} \overline{\mathcal{N}}^i {}_{\mathcal{B}_1^{\text{op}}} = {}^{\mathcal{B}_1} \mathcal{N}_{\mathcal{B}_2}^i \boxtimes {}^{\mathcal{B}_2} \mathcal{P}_{\mathcal{B}_1}^i \end{aligned}$$

□

Lemma 6.4. *Equation (6.2) holds.*

Proof. In view of Lemma 6.2 and the invertibility of \mathcal{K} , it suffices to show that

$$\mathcal{P}^j \boxtimes \mathcal{P}_i \sim \mathcal{P}^i \boxtimes \mathcal{P}_j.$$

Note that generators for both bimodules in Equation (6.2) correspond to partial Kauffman states in the partial knot diagrams containing the two crossings. Clearly,

these generators are independent of the order; to see that the bimodules are independent of the order, it suffices to construct a third, more symmetric bimodule (where we think of the two crossings as appearing in the same level), that is quasi-isomorphic to both.

Generators for this bimodule will also correspond to partial Kauffman states. There will in general be 16 generator types, labelled by pairs of letters among **N**, **S**, **W**, and **E**, corresponding to the four local choices at each of the two crossings. The computation is mostly straightforward. In the special case where the two crossings are adjacent, there are only 15 generator types: there are no generators of type **EW**. This occurs, for example, when $i = 1$ and $j = 3$; see Figure 23. In this case, the DD bimodule has the following local terms:

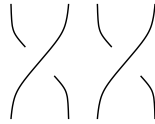
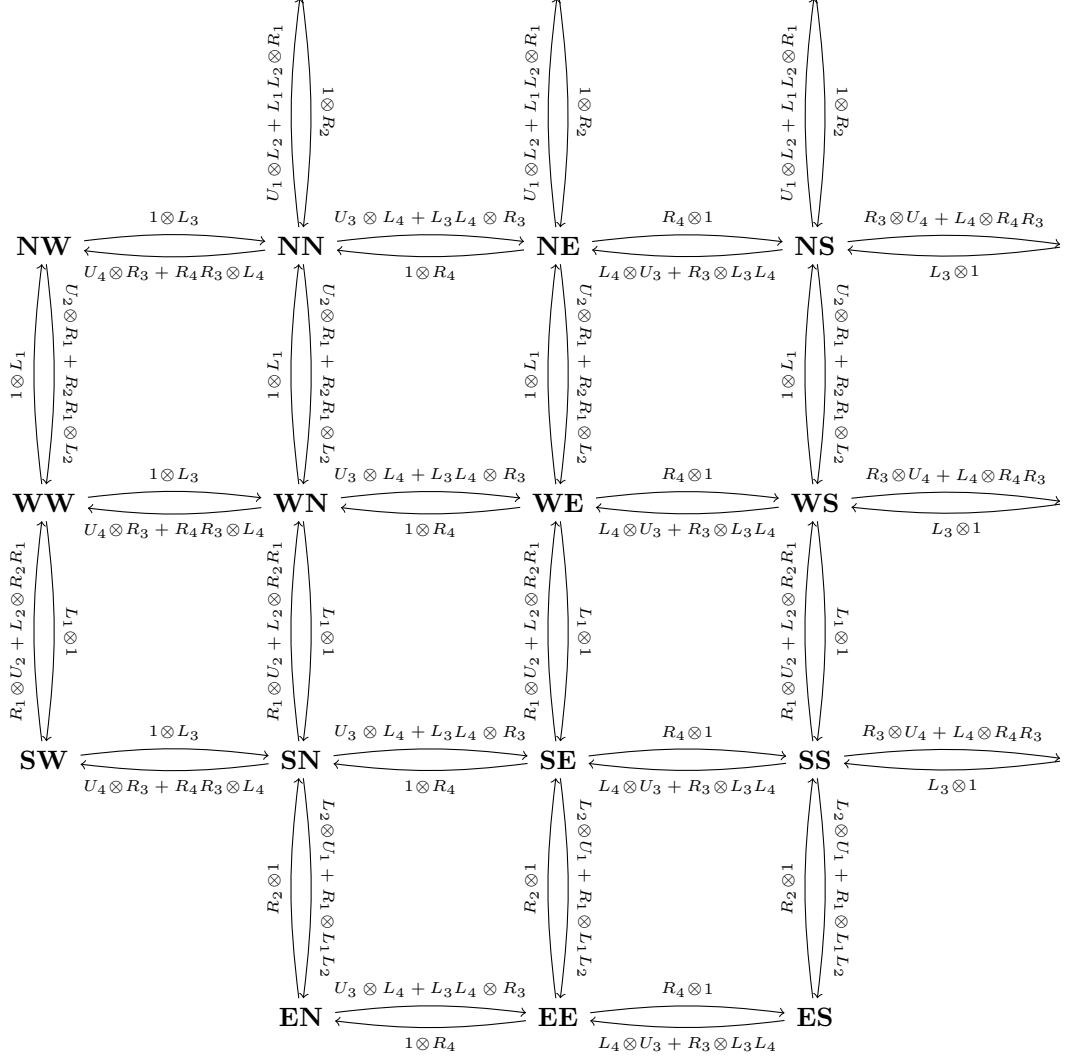


FIGURE 23. When the two crossings appearing in Equation (6.2) are adjacent, there are only 15 types of partial Kauffman states; a partial Kauffman state cannot associate **E** to the crossing on the left and **W** to the one on the right.



As usual, we have suppressed here the distant arrows and the additional arrows of the form $U_\tau(i) \otimes C_i$ or $C_\tau(i) \otimes U_i$. \square

Lemma 6.5. *Equation (6.3) holds.*

Proof. We describe the case where $i = 1$. We first compute ${}^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_2}^2 \boxtimes {}^{\mathcal{B}_2, \mathcal{B}'_1} \mathcal{P}_1$. Again, generators correspond to partial Kauffman states, which we now label as an ordered pair, showing which crossing is associated to which region; see Figure 24. When $\{1, 2, 3\} \subset \mathcal{S}$, after cancelling arrows, we find that the bimodule is given by

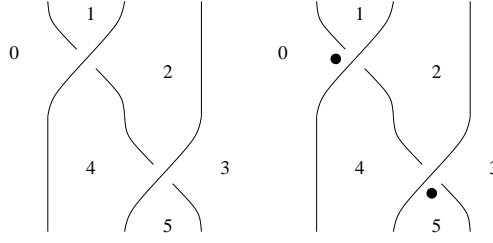
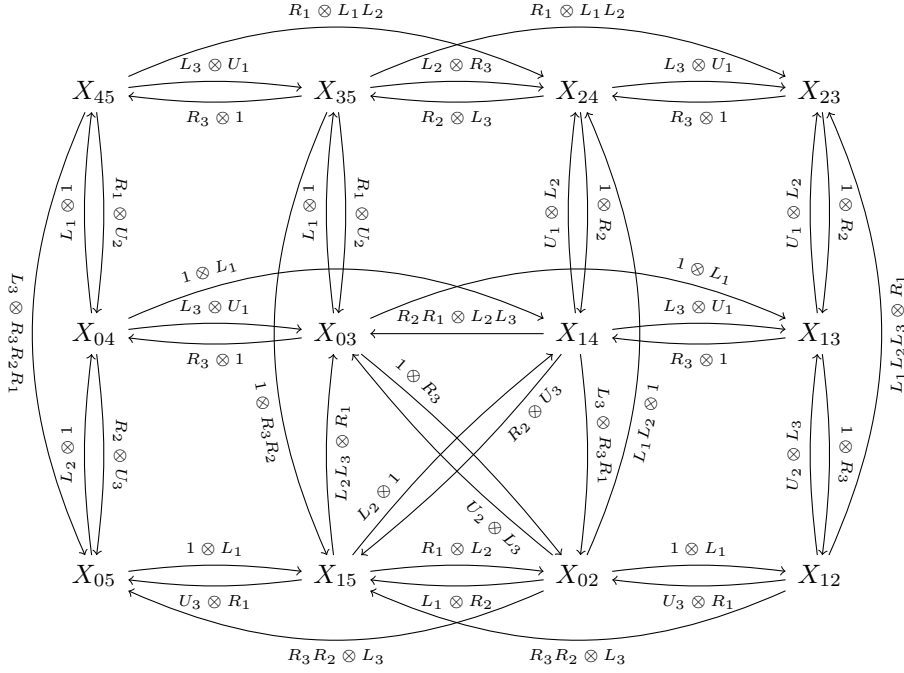


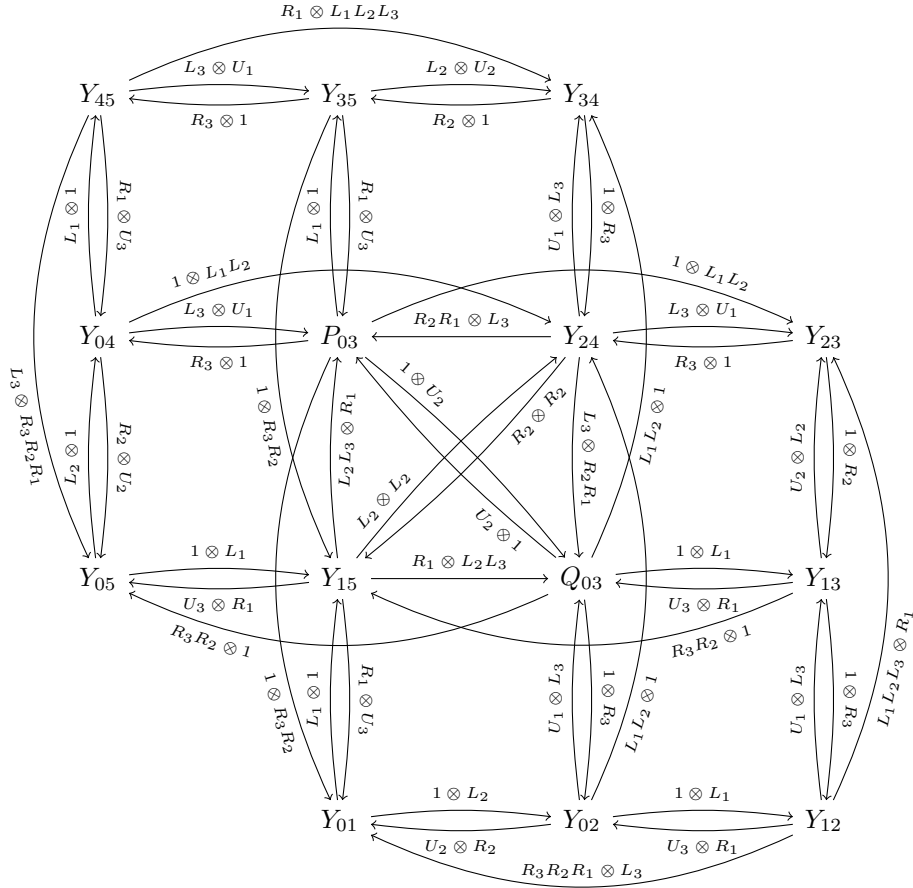
FIGURE 24. At the right, we have shown a generator of type SW ; or equivalently, X_{05} .



along with the usual outside arrows and self-arrows of the form $C_i \otimes U_{\tau_2 \tau_1(i)}$ or $U_i \otimes C_{\tau_2 \tau_1(i)}$, depending on whether or not $i \in \mathcal{S}$. When $\{1, 2, 3\} \cap \mathcal{S}$ is a proper subset of $\{1, 2, 3\}$, we need to apply additional homotopies to obtain this bimodule, as in the proof of Lemma 6.2.

Tensoring this on the left with $\mathcal{B}_5 \mathcal{P}_{\mathcal{B}_3}^i$, and following the labelling conventions from Figure 25, we find that the bimodule is homotopic to one of the following form:

(6.7)



along with the usual outside arrows and self-arrows.

Tensoring in the other order gives the same bimodule; this can be seen quickly from a symmetry on the answer. The picture after the Reidemeister move can be realized as a rotation by 180° (i.e. rotate the leftmost picture in Figure 25). There is a corresponding action on the algebra, exchanging the two tensor factors of the algebra, and further exchanging R_i and L_{4-i} . The corresponding symmetry of the bimodule can be realized by rotating the description from Equation (6.7) by 180° . The fact that the bimodule is fixed by the symmetry implies the claimed invariance under Reidemeister 3 moves. \square

6.1. Other symmetries in the crossing bimodule. The canonical DD bimodules commute with the action of the braid group, in the following sense.

Lemma 6.6. *Fix $0 \leq k \leq m+1$ and an arbitrary subset $\mathcal{S} \subset \{1, \dots, m\}$, and let*

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B}(m, k, \mathcal{S}), & \mathcal{B}_2 &= \mathcal{B}(m, k, \tau_i(\mathcal{S})) \\ \mathcal{B}'_1 &= \mathcal{B}(m, m+1-k, \{1, \dots, m\} \setminus \mathcal{S}) & \mathcal{B}'_2 &= \mathcal{B}(m, m+1-k, \{1, \dots, m\} \setminus \tau_i(\mathcal{S})) \end{aligned}$$

There is an equivalence

$$\mathcal{B}_2 \mathcal{P}_{\mathcal{B}_1}^i \boxtimes \mathcal{B}_1, \mathcal{B}'_1 \mathcal{K} \simeq \mathcal{B}'_1 \mathcal{P}_{\mathcal{B}'_2}^i \boxtimes \mathcal{B}'_2, \mathcal{B}_2 \mathcal{K}.$$

Proof. This is an immediate consequence of Lemma 6.2, and the symmetry of the \mathcal{P}_i from Equation (4.3). \square

7. THE DD BIMODULE OF A CRITICAL POINT

In Sections 8 and 9, we will construct DA bimodules ${}^{\mathcal{B}_2}\Omega_{{\mathcal{B}}'_1}$ and ${}^{\mathcal{B}_1}\mathcal{U}_{{\mathcal{B}}'_2}$ (where the algebras will be made precise shortly) associated to a region in the knot diagram where there are no crossings and a single critical point, which can be a maximum (as in Section 8) or a minimum (as in Section 9). In the present section, we will construct a type DD bimodule, called *the type DD bimodule for a critical point*, ${}^{\mathcal{B}_1, \mathcal{B}_2}\mathcal{E}$ which is related to the aforementioned type DA bimodules via equivalences

$${}^{\mathcal{B}_2}\Omega_{{\mathcal{B}}'_1} \boxtimes {}^{\mathcal{B}'_1, \mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}_1}\mathcal{E} \quad \text{and} \quad {}^{\mathcal{B}_2}\mathcal{U}_{{\mathcal{B}}'_1} \boxtimes {}^{\mathcal{B}'_1, \mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}_1}\mathcal{E}.$$

(See Propositions 8.4 and 9.5 for precise statements.)

Fix integers c , k , and m with $1 \leq c \leq m+1$ and $0 \leq k \leq m+1$. The algebras appearing in the bimodule for a critical point are specified as follows. Let $\phi_c: \{1, \dots, m\} \rightarrow \{1, \dots, m+2\}$ be the function

$$(7.1) \quad \phi_c(j) = \begin{cases} j & \text{if } j < c \\ j+2 & \text{if } j \geq c. \end{cases}$$

Let

$$\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}_1) \quad \text{and} \quad \mathcal{B}_2 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_2),$$

where $\mathcal{S}_1 \subset \{1, \dots, m\}$ and $\mathcal{S}_2 \subset \{1, \dots, m+2\}$, so that

$$\phi_c(\mathcal{S}_1) \cap \mathcal{S}_2 = \emptyset \quad \text{and} \quad |\mathcal{S}_1| + |\mathcal{S}_2| = m+1 \quad \text{and} \quad |\mathcal{S}_2 \cap \{c, c+1\}| = 1;$$

equivalently,

$$\mathcal{S}_2 = \phi_c(\{1, \dots, m\} \setminus \mathcal{S}_1) \cup \{c\} \quad \text{or} \quad \mathcal{S}_2 = \phi_c(\{1, \dots, m\} \setminus \mathcal{S}_1) \cup \{c+1\}.$$

In this section, we construct the type DD bimodule for a critical point, denoted $\mathcal{E}_c = {}^{\mathcal{B}_1, \mathcal{B}_2}\mathcal{E}_c$, where here c (which we will sometimes drop from the notation) indicates where the critical point occurs.

Having specified the algebras, we now describe the underlying vector space for \mathcal{E}_c . We call an idempotent state \mathbf{y} for \mathcal{B}_2 an *allowed idempotent state for \mathcal{B}_2* if

$$|\mathbf{y} \cap \{c-1, c, c+1\}| \leq 2 \quad \text{and} \quad c \in \mathbf{y}.$$

There is a map ψ from allowed idempotent states \mathbf{y} for \mathcal{B}_2 to idempotent states for \mathcal{B}_1 , where $\mathbf{x} = \psi(\mathbf{y}) \subset \{0, \dots, m\}$ is characterized by

$$(7.2) \quad |\mathbf{y} \cap \{c-1, c, c+1\}| + |\mathbf{x} \cap \{c-1\}| = 2 \quad \text{and} \quad \phi_c(\mathbf{x}) \cap \mathbf{y} = \emptyset.$$

As a vector space, \mathcal{E}_c is spanned by vectors that are in one-to-one correspondence with allowed idempotent states for \mathcal{B}_2 . The bimodule structure, over the rings of idempotents $\mathbf{I}(\mathcal{B}_1)$ and $\mathbf{I}(\mathcal{B}_2)$, is specified as follows. If $\mathbf{P} = \mathbf{P}_{\mathbf{y}}$ is the generator associated to the idempotent state \mathbf{y} , then for idempotent states \mathbf{x} and \mathbf{z} for \mathcal{B}_1 and \mathcal{B}_2 respectively,

$$(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{z}}) \cdot \mathbf{P}_{\mathbf{y}} = \begin{cases} \mathbf{P}_{\mathbf{y}} & \text{if } \mathbf{y} = \mathbf{z} \text{ and } \mathbf{x} = \psi(\mathbf{y}) \\ 0 & \text{otherwise.} \end{cases}.$$

To specify the differential, consider the element $A \in \mathcal{B}_1 \otimes \mathcal{B}_2$

$$(7.3) \quad A = (1 \otimes L_c L_{c+1}) + (1 \otimes R_{c+1} R_c) + \sum_{j=1}^m R_j \otimes L_{\phi(j)} + L_j \otimes R_{\phi(j)} \\ + \left\{ \begin{array}{ll} (1 \otimes C_c U_{c+1}) & \text{if } c \in \mathcal{S}_2 \\ (1 \otimes U_c C_{c+1}) & \text{if } c+1 \in \mathcal{S}_2, \end{array} \right\} + \sum_{j=1}^m \left\{ \begin{array}{ll} C_j \otimes U_{\phi(j)} & \text{if } j \in \mathcal{S}_1 \\ U_j \otimes C_{\phi(j)} & \text{if } j \notin \mathcal{S}_1 \end{array} \right\}.$$

where we have dropped the subscript c from $\phi_c = \phi$. Let

$$\delta^1(\mathbf{P}_{\mathbf{y}}) = (\mathbf{I}_{\psi(\mathbf{y})} \otimes \mathbf{I}_{\mathbf{y}}) \cdot A \otimes \sum_{\mathbf{z}} \mathbf{P}_{\mathbf{z}},$$

where the latter sum is taken over all allowed idempotent states \mathbf{z} for \mathcal{B}_2 .

Lemma 7.1. *The space ${}^{\mathcal{B}_1, \mathcal{B}_2} \mathcal{E}_c$ defined above, and equipped with the map*

$$\delta^1: \mathcal{E}_c \rightarrow (\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{E}_c,$$

specified above, is a type DD bimodule over \mathcal{B}_1 and \mathcal{B}_2 .

Proof. The proof is a straightforward adaptation of Lemma 3.9. \square

It is helpful to understand \mathcal{E}_c a little more explicitly. To this end, we classify the allowed idempotents for \mathcal{B}_2 into three types, labelled **X**, **Y**, and **Z**:

- **y** is of type **X** if $\mathbf{y} \cap \{c-1, c, c+1\} = \{c-1, c\}$,
- **y** is of type **Y** if $\mathbf{y} \cap \{c-1, c, c+1\} = \{c, c+1\}$,
- **y** is of type **Z** if $\mathbf{y} \cap \{c-1, c, c+1\} = \{c\}$.

There is a corresponding classification of the generators $\mathbf{P}_{\mathbf{y}}$ into **X**, **Y**, and **Z**, according to the type of **y**; see Figure 26.

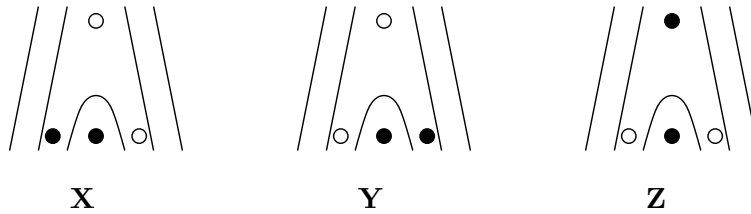


FIGURE 26. **DD bimodule of a critical point.** Three generator types are illustrated.

With respect to this decomposition, terms in the differential are of the following four types:

- (P-1) $R_j \otimes L_{\phi(j)}$ and $L_j \otimes R_{\phi(j)}$ for all $j \in \{1, \dots, m\} \setminus \{c-1, c\}$; these connect generators of the same type.
- (P-2) $C_j \otimes U_{\phi(j)}$ if $j \in \mathcal{S}_1$ and $U_j \otimes C_{\phi(j)}$ if $j \in \{1, \dots, m\} \setminus \mathcal{S}_1$
- (P-3) $1 \otimes U_c C_{c+1}$ if $c+1 \in \mathcal{S}_2$ or $1 \otimes C_c U_{c+1}$ if $c \in \mathcal{S}_2$.

(P-4) Terms in the diagram below connect generators of different types.

$$(7.4) \quad \begin{array}{ccc} & & \\ & \xrightarrow{1 \otimes L_c L_{c+1}} & \\ \mathbf{X} & & \mathbf{Y} \\ & \xleftarrow{1 \otimes R_{c+1} R_c} & \\ & & \\ & \swarrow L_{c-1} \oplus R_{c-1} \quad \searrow L_c \oplus R_{c+2} & \\ & \mathbf{Z} & \\ & \nwarrow R_{c-1} \oplus L_{c-1} \quad \nearrow R_c \oplus L_{c+2} & \end{array}$$

With the understanding that if $c = 1$, then the terms containing L_{c-1} or R_{c-1} are missing; similarly, if $c = m + 1$, the terms containing R_{c+2} and L_{c+2} are missing.

7.1. Grading sets. For any generator P of \mathcal{E}_c , if $(a_2 \otimes a_1) \otimes Q$ appears with non-zero multiplicity in $d^1(P)$, then for all $i = 1, \dots, m$, $w_i(a_1) = w_{\phi_c(i)}(a_2)$ and $w_c(a_2) = w_{c+1}(a_2)$. This is obvious from the form of the bimodule (Equation (7.3)). In the language of Section 3.9, we can express this by saying that the bimodule is supported in grading 0 in $H_1(W, \partial W)$, where W is the partial knot diagram with a single critical point in it.

7.2. Critical points and crossings. Later, we will study in detail how the bimodules of critical points interact with the bimodules associated to crossings. For the time being, we will content ourselves with the following special case, where the critical point and the crossing are connected to each other.

To set notation, let $\psi_c: \{1, \dots, m\} \rightarrow \{1, \dots, m+2\}$ be the function

$$\psi_c(j) = \begin{cases} j & \text{if } j < c \\ c+1 & \text{if } j = c \\ j+2 & \text{if } j > c; \end{cases}$$

i.e. $\psi_c = \tau_{c+1} \circ \phi_c = \tau_c \phi_{c+1}$.

Lemma 7.2. Fix integers $0 \leq k \leq m+1$, and an arbitrary subset $\mathcal{S}_1 \subset \{1, \dots, m\}$, and let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}_1)$, $\mathcal{B}_2 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_2)$, $\mathcal{B}_3 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_3)$, $\mathcal{B}_4 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_4)$ where

$\mathcal{S}_2 = \phi_{c+1}(\{1, \dots, m\} \setminus \mathcal{S}_1) \cup \{c+1\}$ or $\mathcal{S}_2 = \phi_{c+1}(\{1, \dots, m\} \setminus \mathcal{S}_1) \cup \{c+2\}$, $\mathcal{S}_3 = \tau_c(\mathcal{S}_2)$ and $\mathcal{S}_4 = \tau_{c+1}(\mathcal{S}_3)$. There is homotopy equivalence of graded bimodules:

$$(7.5) \quad {}^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_2}^c \boxtimes {}^{\mathcal{B}_2, \mathcal{B}_1} \mathcal{E}_{c+1} \simeq {}^{\mathcal{B}_3} \mathcal{N}_{\mathcal{B}_4}^{c+1} \boxtimes {}^{\mathcal{B}_4, \mathcal{B}_1} \mathcal{E}_c$$

Remark 7.3. The gradings on the crossing bimodules were chosen so that Equation (7.5) holds as a bigraded module map.

The above lemma is a straightforward computation, using the following *trident bimodule* ${}^{\mathcal{B}_1, \mathcal{B}_3} \mathcal{T}$.

Generators correspond to pairs of idempotents \mathbf{x} and \mathbf{y} for \mathcal{B}_1 and \mathcal{B}_3 with the following properties:

- Letting, then $\psi_c(\mathbf{x}) \cap \mathbf{y} = \emptyset$.
- $|\mathbf{y}| = |\mathbf{x}| + 1$
- $|\mathbf{y} \cap \{c, c+1\}| \geq 1$
- if $|\mathbf{y} \cap \{c, c+1\}| = 2$ then either $c-1 \notin \mathbf{x}$ and $c-1 \notin \mathbf{y}$ or $c \notin \mathbf{x}$ and $c+2 \notin \mathbf{y}$

We separate these pairs into four types:

- (\mathbf{x}, \mathbf{y}) is of Type **P** if $c-1 \notin \mathbf{x}$ and $c-1 \notin \mathbf{y}$ (so $\{c, c+1\} \subset \mathbf{y}$)
- (\mathbf{x}, \mathbf{y}) is of Type **Q** if $c \notin \mathbf{x}$ and $c+2 \notin \mathbf{y}$ (so $\{c, c+1\} \subset \mathbf{y}$)
- (\mathbf{x}, \mathbf{y}) is of Type **X** if $c \notin \mathbf{y}$
- (\mathbf{x}, \mathbf{y}) is of Type **Y** if $c+1 \notin \mathbf{y}$

See Figure 27 for a picture. (Note that these generator types correspond to the partial Kauffman states in the sense of Definition 5.1 of the diagram containing the maximum and the crossing, with the understanding that the incoming idempotent is complementary to idempotent state coming from the top of the diagram.)

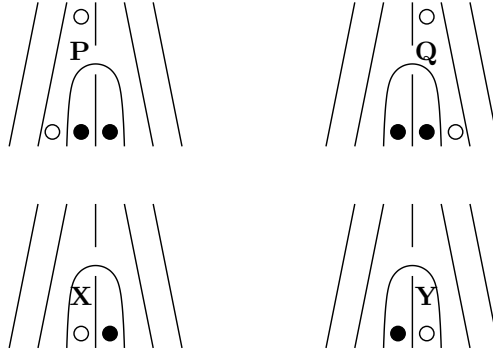


FIGURE 27. **Trident bimodule.** The four generator types are illustrated.

Let $\mathbf{T}_{\mathbf{x}, \mathbf{y}}$ be the corresponding generator of \mathcal{T} . As a left module over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_3)$, the action is specified by $(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) \cdot \mathbf{T}_{\mathbf{x}, \mathbf{y}} = \mathbf{T}_{\mathbf{x}, \mathbf{y}}$. The differential has the following types of terms:

- (1) $R_j \otimes L_j$ and $L_j \otimes R_j$ for all $j \in \{1, \dots, m\} \setminus \{c\}$; these connect generators of the same type.
- (2) $C_j \otimes U_{\psi_c(j)}$ if $j \in \mathcal{S}_1 \setminus \{c\}$ and $U_j \otimes C_{\psi_c(j)}$ if $j \in \{1, \dots, m\} \setminus (\mathcal{S}_1 \cup \{c\})$; these connect generators of the same type.
- (3) $C_c \otimes U_{c+1}$ if $c \in \mathcal{S}_1$ and $U_c \otimes C_{c+1}$ if $c \notin \mathcal{S}_1$

(4) Terms in the diagram below connect generators of different types:

$$\begin{array}{ccc}
 & \xleftrightarrow{L_c \otimes U_{c+1} + 1 \otimes L_c L_{c+1} L_{c+2}} & \\
 \mathbf{P} & \xleftrightarrow{R_c \otimes 1} & \mathbf{Q} \\
 \uparrow \scriptstyle 1 \otimes L_c U_{c+2} + L_c \otimes R_{c+2} R_{c+1} & & \downarrow \scriptstyle 1 \otimes R_{c+2} + L_c \otimes R_{c+1} R_c \\
 & \xleftrightarrow{U_c \otimes R_{c+1} + R_c \otimes L_c L_{c+2}} & \\
 \mathbf{X} & \xleftrightarrow{1 \otimes L_{c+1}} & \mathbf{Y}
 \end{array}$$

(7.6)

Proof of Lemma 7.2. A straightforward computation identifies both sides with the trident bimodule specified above, after a possible homotopy (as in the proof of Lemma 6.2). The fact that the map respects gradings follows quickly from the grading conventions; see Equations (4.5) and (4.7). \square

8. THE DA BIMODULE ASSOCIATED TO A MAXIMUM

We will describe now the type DA bimodule ${}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c$ of a region in the knot diagram where there are no crossings and a single local maximum, which occurs between the c^{th} and the $(c+1)^{st}$ outgoing strand.

Specifically, fix integers c , k , and m with $1 \leq c \leq m+1$ and $0 \leq k \leq m+1$. Let

$$\mathcal{B}'_1 = \mathcal{B}(m, k, \mathcal{S}_1) \quad \text{and} \quad \mathcal{B}_2 = \mathcal{B}(m+2, k+1, \mathcal{S}_2),$$

where $\mathcal{S}'_1 \subset \{1, \dots, m\}$ is arbitrary, and $\mathcal{S}_2 \subset \{1, \dots, m+2\}$ is given by

$$\mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c\} \text{ or } \mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c+1\}.$$

The two cases as corresponding to the two possible orientations of the strand with the maximum; see Figure 28.

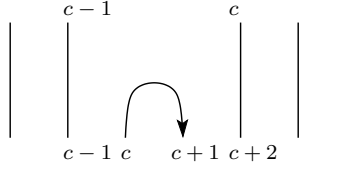


FIGURE 28. **Picture of maximum.** Here the maximum is oriented from left to right, so $\mathcal{S}_2 \cap \{c, c+1\} = c$.

As in Section 7, an *allowed idempotent state* for \mathcal{B}_2 is an idempotent state \mathbf{y} for \mathcal{B}_2 with

$$c \in \mathbf{y} \quad \text{and} \quad |\mathbf{y} \cap \{c-1, c+1\}| \leq 1$$

There is a map ψ' from allowed idempotent states for \mathcal{B}_2 and idempotent states for \mathcal{B}'_1 , given by

$$\psi'(\mathbf{x}) = \begin{cases} \phi^{-1}(\mathbf{y}) & \text{if } c+1 \notin \mathbf{y} \\ \phi^{-1}(\mathbf{y}) \cup \{c-1\} & \text{if } c+1 \in \mathbf{y} \end{cases}$$

Observe that $\psi'(\mathbf{y}) = \{0, \dots, m\} \setminus \psi(\mathbf{y})$, where ψ is the map from Section 7 (c.f. Equation (7.2)).

A basis for the underlying vector space of ${}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c$ is specified by the allowed idempotent states for \mathcal{B}_2 . The bimodule structure, over the rings of idempotents $\mathbf{I}(\mathcal{B}'_1)$ and $\mathbf{I}(\mathcal{B}_2)$, is specified as follows. If $\mathbf{Q}_{\mathbf{y}}$ is the generator associated to the allowed idempotent state \mathbf{y} , then for idempotent states \mathbf{x} and \mathbf{z} for \mathcal{B}'_1 and \mathcal{B}_2 respectively,

$$\mathbf{I}_{\mathbf{z}} \cdot \mathbf{Q}_{\mathbf{y}} \cdot \mathbf{I}_{\mathbf{x}} = \begin{cases} \mathbf{Q}_{\mathbf{y}} & \text{if } \mathbf{y} = \mathbf{z} \text{ and } \mathbf{x} = \psi'(\mathbf{y}) \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$\delta_1^1: {}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c \rightarrow \mathcal{B}_2 \otimes_{\mathbf{I}(\mathcal{B}_2)} {}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c$$

is given by

$$\mathbf{Q}_{\mathbf{y}} \mapsto \mathbf{I}_{\mathbf{y}} \cdot \left(R_{c+1}R_c + L_cL_{c+1} + \begin{cases} U_cC_{c+1} & \text{if } c+1 \in \mathcal{S}_2 \\ C_cU_{c+1} & \text{if } c \in \mathcal{S}_2, \end{cases} \right) \otimes \sum_{\mathbf{z}} \mathbf{Q}_{\mathbf{z}}.$$

where the sum is taken over all allowed idempotents \mathbf{z} for \mathcal{B}_2 .

We split the bimodule

$$\mathcal{B}_2 \Omega_{\mathcal{B}'_1}^c \cong \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$$

according to the types of the corresponding idempotents as defined in Section 7; i.e.

$$\begin{aligned} \mathbf{X} &= \bigoplus_{\{\mathbf{y} \mid \mathbf{y} \cap \{c-1, c, c+1\} = \{c-1, c\}\}} \mathbf{Q}_{\mathbf{y}} \\ \mathbf{Y} &= \bigoplus_{\{\mathbf{y} \mid \mathbf{y} \cap \{c-1, c, c+1\} = \{c, c+1\}\}} \mathbf{Q}_{\mathbf{y}} \\ \mathbf{Z} &= \bigoplus_{\{\mathbf{y} \mid \mathbf{y} \cap \{c-1, c, c+1\} = \{c\}\}} \mathbf{Q}_{\mathbf{y}} \end{aligned}$$

With respect to this splitting, δ_1^1 can be expressed as follows. If $c \in \mathcal{S}_2$, then

$$\begin{aligned} \delta_1^1(\mathbf{X}) &= C_c U_{c+1} \otimes \mathbf{X} + R_{c+1} R_c \otimes \mathbf{Y} \\ \delta_1^1(\mathbf{Y}) &= C_c U_{c+1} \otimes \mathbf{Y} + L_c L_{c+1} \otimes \mathbf{X} \\ \delta_1^1(\mathbf{Z}) &= C_c U_{c+1} \otimes \mathbf{Z}; \end{aligned}$$

whereas if $c+1 \in \mathcal{S}_2$,

$$\begin{aligned} \delta_1^1(\mathbf{X}) &= U_c C_{c+1} \otimes \mathbf{X} + R_{c+1} R_c \otimes \mathbf{Y} \\ \delta_1^1(\mathbf{Y}) &= U_c C_{c+1} \otimes \mathbf{Y} + L_c L_{c+1} \otimes \mathbf{X} \\ \delta_1^1(\mathbf{Z}) &= U_c C_{c+1} \otimes \mathbf{Z}; \end{aligned}$$

To define δ_2^1 , it is helpful to have the following:

Lemma 8.1. *Let \mathbf{x}' and \mathbf{y}' be two idempotents for \mathcal{B}'_1 that are close enough, and \mathbf{x} be an allowed idempotent state for \mathcal{B}_2 with $\psi'(\mathbf{x}) = \mathbf{x}'$. Then, there is a uniquely associated allowed idempotent state \mathbf{y} with $\psi'(\mathbf{y}) = \mathbf{y}'$ so that there is a surjective map*

$$\Phi_{\mathbf{x}}: \mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{y}'} \rightarrow \mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{y}}$$

that maps the portion of $\mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{y}'}$ with weights (w'_1, \dots, w'_m) surjectively onto the portion of $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{y}}$ with $w_{\phi_c(i)} = w'_i$ and $w_c = w_{c+1} = 0$, and that satisfies the relations

$$\Phi_{\mathbf{x}}(U_i \cdot a) = U_{\phi_c(i)} \cdot \Phi_{\mathbf{x}}(a) \quad \text{and} \quad \Phi_{\mathbf{x}}(C_j \cdot a) = C_{\phi(j)} \cdot \Phi_{\mathbf{x}}(a)$$

for any $i \in 1, \dots, m$, $j \in \mathcal{S}'_1$, and $a \in \mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{y}'}$.

Proof. Recall the weight $v^{\mathbf{x}}$ of idempotents, as defined in Equation (3.1). Since \mathbf{x}' and \mathbf{y}' are close enough, then there is some integer $j \in \mathbb{Z}$ so that exactly one of the following holds:

- (1) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j, j-1)$
- (2) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j, j)$
- (3) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j+1, j)$
- (4) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j-1)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j-1, j-2)$
- (5) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j-1)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j-1, j-1)$

- (6) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j-1)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j, j-1)$
- (7) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j-1)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j, j)$
- (8) $(v_{c-1}^{\mathbf{x}'}, v_c^{\mathbf{x}'}) = (j, j-1)$ and $(v_{c-1}^{\mathbf{y}'}, v_c^{\mathbf{y}'}) = (j+1, j)$

In each of these cases, we claim that there is a unique allowed idempotent state \mathbf{y} for \mathcal{B}_2 with $\psi'(\mathbf{y}) = \mathbf{y}'$ so that

$$(8.1) \quad (v_{c-1}^{\mathbf{y}}, v_c^{\mathbf{y}}) = (v_{c-1}^{\mathbf{x}}, v_c^{\mathbf{x}}).$$

To specify \mathbf{y} , it now suffices to specify its type, as we do in the eight cases listed above:

- (1) \mathbf{x} is of type \mathbf{Z} ; \mathbf{y} is of type \mathbf{Y} .
- (2) \mathbf{x} is of type \mathbf{Z} and \mathbf{y} is of type \mathbf{Z}
- (3) \mathbf{x} is of type \mathbf{Z} ; \mathbf{y} is of type \mathbf{X} .
- (4) \mathbf{x} is of type \mathbf{X} and \mathbf{y} is of type \mathbf{Y} .
- (5) \mathbf{x} is of type \mathbf{X} and \mathbf{y} is of type \mathbf{Z}
- (6) \mathbf{x} and \mathbf{y} are both of type \mathbf{X} or both of type \mathbf{Y}
- (7) \mathbf{x} is of type \mathbf{Y} and \mathbf{y} is of type \mathbf{Z}
- (8) \mathbf{x} is of type \mathbf{Y} and \mathbf{y} is of type \mathbf{X} .

Recall that there are graded identifications

$$\begin{aligned} \phi^{\mathbf{x}', \mathbf{y}'} : \mathbb{F}[U_1, \dots, U_m] &\rightarrow \mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}'} \\ \phi^{\mathbf{x}, \mathbf{y}} : \mathbb{F}[U_1, \dots, U_{m+2}] &\rightarrow \mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m+2, k+1) \cdot \mathbf{I}_{\mathbf{y}}. \end{aligned}$$

The ring map

$$\varphi : \mathbb{F}[U_1, \dots, U_m] \rightarrow \mathbb{F}[U_1, \dots, U_{m+2}]$$

with $\varphi(1) = 1$ and $\varphi(U_i) = U_{\phi(i)}$ induces a map

$$\mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}'} \rightarrow \mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m+2, k+1) \cdot \mathbf{I}_{\mathbf{y}}$$

that maps the portion of $\mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_0(m, k) \cdot \mathbf{I}_{\mathbf{y}'}$ with weights fixed at (w'_1, \dots, w'_m) onto the portion of $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_0(m+2, k+1) \cdot \mathbf{I}_{\mathbf{y}}$ with weights fixed at $w_i = w'_{\phi(i)}$ and $w_c = w_{c+1} = 0$.

In the eight above cases, we check that $\varphi(\mathcal{I}(\mathbf{x}', \mathbf{y}'))$ is mapped into $\mathcal{I}(\mathbf{x}, \mathbf{y})$. In all cases other than Case 6, there are no generating intervals for $(\mathbf{x}', \mathbf{y}')$ that contain $c-1$ in their interior. Moreover, the images of these generating intervals under ϕ_c are generating intervals for (\mathbf{x}, \mathbf{y}) . In Case 6, there is a generating interval $[p, q]$ with $p < c-1 < q$. If, furthermore, \mathbf{x} and \mathbf{y} are of type \mathbf{X} , then $[c+2, \dots, q+2]$ is a generating interval for (\mathbf{x}, \mathbf{y}) , so $\mathcal{I}(\mathbf{x}', \mathbf{y}')$ is still mapped into $\mathcal{I}(\mathbf{x}, \mathbf{y})$. Similarly, if \mathbf{x} and \mathbf{y} are of type \mathbf{Y} , $[p, \dots, c-1]$ is a generating interval for (\mathbf{x}, \mathbf{y}) , so once again $\mathcal{I}(\mathbf{x}', \mathbf{y}')$ is mapped into $\mathcal{I}(\mathbf{x}, \mathbf{y})$.

It follows that φ induces the map $\Phi_{\mathbf{x}}$ on $\mathcal{B}(m, k) \subset \mathcal{B}'_1$, which we can extend so that $\Phi_x(a \cdot C_j) = C_{\phi(j)} \cdot \Phi_{\mathbf{x}}(a)$ for all $j \in \mathcal{S}'_1$ to get the map required by the lemma. \square

Lemma 8.2. *Suppose that \mathbf{x}'_1 , \mathbf{x}'_2 , and \mathbf{x}'_3 are three idempotent states for \mathcal{B}'_1 , so that \mathbf{x}'_i is close enough to \mathbf{x}'_{i+1} for $i = 1, 2$; and choose any \mathbf{x}_1 so that $\psi(\mathbf{x}_1) = \mathbf{x}'_1$. Let \mathbf{x}_2 be the idempotent state associated to $(\mathbf{x}_1, \mathbf{x}'_1, \mathbf{x}'_2)$ as in Lemma 8.1, and*

then \mathbf{x}_3 be associated to $(\mathbf{x}_2, \mathbf{x}'_2, \mathbf{x}'_3)$ by that lemma. Then, there is a commutative diagram:

$$\begin{array}{ccc} (\mathbf{I}_{\mathbf{x}'_1} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{x}'_2}) \otimes (\mathbf{I}_{\mathbf{x}'_2} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{x}'_3}) & \xrightarrow{\mu'_2} & \mathbf{I}_{\mathbf{x}'_1} \cdot \mathcal{B}'_1 \cdot \mathbf{I}_{\mathbf{x}'_3} \\ \Phi_{\mathbf{x}_1} \otimes \Phi_{\mathbf{x}_2} \downarrow & & \downarrow \Phi_{\mathbf{x}_1} \\ (\mathbf{I}_{\mathbf{x}_1} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{x}_2}) \otimes (\mathbf{I}_{\mathbf{x}_2} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{x}_3}) & \xrightarrow{\mu_2} & \mathbf{I}_{\mathbf{x}_1} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{x}_3}, \end{array}$$

where μ'_2 and μ_2 are multiplications on \mathcal{B}'_1 and \mathcal{B}_2 respectively.

Proof. Commutativity of this diagram is an immediate consequence of the fact that Φ preserves the weights, which in turn determine the multiplication on the algebras. \square

If $a = \mathbf{I}_{\mathbf{x}'} \cdot a \cdot \mathbf{I}_{\mathbf{y}'} \in \mathcal{B}(m, k) \subset \mathcal{B}'_1$ is a non-zero idempotent, and \mathbf{x} is any allowed idempotent with $\psi'(\mathbf{x}) = \mathbf{x}'$, let

$$\delta_2^1(\mathbf{Q}_{\mathbf{x}}, a) = \Phi_{\mathbf{x}}(a) \otimes \mathbf{Q}_{\mathbf{y}},$$

where \mathbf{y} is the idempotent from Lemma 8.1 associated to $(\mathbf{x}, \mathbf{x}', \mathbf{y}')$; i.e. $\Phi_{\mathbf{x}}(a) \cdot \mathbf{I}_{\mathbf{y}} = \Phi_{\mathbf{x}}(a)$.

Theorem 8.3. *These actions give ${}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c$ the structure of a DA bimodule that is adapted to the corresponding partial knot diagram (containing a single maximum).*

Proof. We consider the case where $c+1 \in \mathcal{S}_2$; the case where $c \in \mathcal{S}_2$ works similarly.

The \mathcal{A}_{∞} relation with no incoming algebra elements has the form

$$(8.2) \quad \begin{array}{c} \downarrow \\ \delta_1^1 \\ \swarrow \downarrow \\ \mu_2^{\mathcal{B}_2} \downarrow \end{array} + \begin{array}{c} \downarrow \\ \delta_1^1 \\ \swarrow \downarrow \\ \mu_1^{\mathcal{B}_2} \downarrow \end{array} = 0.$$

The terms that change type from \mathbf{X} to \mathbf{Y} and back to \mathbf{X} (contributing on the left to Equation (8.2)) contribute

$$(R_{c+1} \cdot R_c \cdot L_c \cdot L_{c+1}) \otimes \mathbf{X} = U_c \cdot U_{c+1} \otimes \mathbf{X};$$

which cancels with $d(C_{c+1} \cdot U_c) \otimes \mathbf{X}$ (contributing on the right to Equation (8.2)). We get a similar argument for generators of type \mathbf{Y} . Since $U_c U_{c+1} \otimes \mathbf{Z} = 0$, Equation (8.2) now follows.

The \mathcal{A}_∞ relation with one algebra input follows from the equations

$$(8.3) \quad \begin{array}{c} \downarrow \\ \delta_2^1 \\ \downarrow \\ \delta_1^1 \\ \swarrow \mu_2^{\mathcal{B}_2} \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \delta_2^1 \\ \downarrow \\ \delta_1^1 \\ \swarrow \mu_2^{\mathcal{B}_2} \\ \downarrow \end{array} = 0$$

and

$$(8.4) \quad \begin{array}{c} \downarrow \\ \delta_2^1 \\ \swarrow \mu_1^{\mathcal{B}_2} \\ \downarrow \end{array} + \begin{array}{c} \downarrow \\ \mu_1^{\mathcal{B}'_1} \\ \downarrow \\ \delta_1^1 \\ \swarrow \mu_1^{\mathcal{B}_2} \\ \downarrow \end{array} = 0.$$

We verify Equation (8.3) for incoming algebra elements $a = \mathbf{I}_{\mathbf{X}'} \cdot a \cdot \mathbf{I}_{\mathbf{Y}'}$. For the terms in δ_1^1 that preserve type, $U_c C_{c+1}$, both have the same contribution: multiplication by $U_c \cdot C_{c+1}$ induces an endomorphism of $\mathbf{I}_{\mathbf{X}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{Y}}$ that maps the portion with $w_c = w_{c+1} = 0$ injectively into $w_c = w_{c+1} = 1$ (since the generating interval for (\mathbf{x}, \mathbf{y}) containing c or $c+1$ contains both c and $c+1$).

Consider next the terms $(L_c L_{c+1}$ and $R_{c+1} R_c)$ where δ_1^1 changes the type of the idempotent. When δ_2^1 changes the type of the idempotent, as well, both terms in Equation (8.3) vanish, since in that case the outgoing algebra element moves too far. Finally, consider the case where δ_2^1 preserves the idempotent, and suppose the incoming generator is of the form $Q_{\mathbf{x}_1}$ where \mathbf{x}_1 is of type \mathbf{X} and $\psi'(\mathbf{x}_1) = \mathbf{x}$. Let \mathbf{x}_2 be the allowed algebra element of type \mathbf{Y} with $\psi'(\mathbf{x}_2) = \mathbf{y}$, so that $\mathbf{I}_{\mathbf{x}_1} \cdot L_c L_{c+1} = L_c L_{c+1} \cdot \mathbf{I}_{\mathbf{x}_2}$. The cancellation of the corresponding terms in Equation (8.3) now follows from the easily verified identity

$$\Phi_{\mathbf{x}_1}(a) \cdot L_c L_{c+1} = L_c L_{c+1} \cdot \Phi_{\mathbf{x}_2}(a).$$

The case where the incoming generator is of type \mathbf{Y} follows from the similar equation:

$$\Phi_{\mathbf{x}_2}(a) \cdot R_{c+1} R_c = R_{c+1} R_c \cdot \Phi_{\mathbf{x}_1}(a).$$

Equation (8.4) follows from the fact that $C_{\phi_c(j)} \cdot \Phi_{\mathbf{x}} = \Phi_{\mathbf{x}} \cdot C_j$ for all $j \in S_1$.

The relation

is equivalent to Lemma 8.2. Since $\delta_\ell^1 = 0$ for $\ell > 2$, the \mathcal{A}_∞ relations now follow.

To verify the grading, note that all the algebra output in the bimodule satisfy $w_c(b) = w_{c+1}(b)$. Combined with the grading properties from Lemma 8.1, it follows that Ω^c is graded by $H^1(W, \partial W)$, where W is specified by the partial knot diagram, and the module is thought of as supported in grading 0. To verify the Maslov grading, note that all the algebra outputs appearing in δ_1^1 have Maslov grading -1 . We can think of Ω^c as supported in Maslov grading 0. It follows readily that Ω^c is adapted to W , as in Definition 3.18. \square

It is convenient to summarize this as follows: when $c+1 \in \mathcal{S}_2$, the δ_1^1 actions are specified by the following diagram:

and the δ_2^1 actions are specified by the diagram

once we include outside actions $L_i \otimes L_{\phi_c(i)}$, $R_i \otimes R_{\phi(i)}$, $U_i \otimes U_{\phi(i)}$ and $C_j \otimes C_{\phi_c(j)}$ for $i \in \{1, \dots, m\}$ and $j \in \mathcal{S}_1$, with the further understanding that δ_2^1 is extended to be multiplicative in the incoming algebra elements.

Proposition 8.4. Ω^c is dual to the module \mathcal{E}_c from Section 7, in the following sense. Let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}_1)$, where $\mathcal{S}_1 \subset \{1, \dots, m\}$; $\mathcal{B}'_1 = \mathcal{B}(m, m+1-k, \mathcal{S}'_1)$, where $\mathcal{S}'_1 = \{1, \dots, m\} \setminus \mathcal{S}_1$; and $\mathcal{B}_2 = \mathcal{B}(m, m+1-k, \mathcal{S}_2)$, where $\mathcal{S}_2 = \mathcal{S}'_1 \cup \{c\}$ or $\mathcal{S}'_1 \cup \{c+1\}$. Then,

$${}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}^c \boxtimes {}^{\mathcal{B}'_1, \mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}_1}\mathcal{E}_c.$$

Proof. This is straightforward to check using the definitions. \square

8.1. A special case. When there is a single maximum, and no other strands, we define a type D structure over the algebra $\mathcal{B}_2 = \mathcal{B}(2, 1, \{1\})$ or $\mathcal{B}(2, 1, \{2\})$, ${}^{\mathcal{B}_2}\Omega$. This type D structure has one generator \mathbf{Z} with $\mathbf{I}_{\{1\}} \cdot \mathbf{Z} = \mathbf{Z}$.

$$\delta^1(\mathbf{Z}) = C_1 U_2 \otimes \mathbf{Z} \quad \text{or} \quad \delta^1(\mathbf{Z}) = U_1 C_2 \otimes \mathbf{Z},$$

according to how the strand is oriented. Obviously, this can be thought of as a degenerate case of the earlier construction, where the incoming algebra $\mathcal{B}'_1 = \mathcal{B}(0, 0, \emptyset) \cong \mathbb{F}$.

8.2. Partial Kauffman states and grading sets. Formally, the generators can be thought of as corresponding to partial Kauffman states. There are no crossings, and it has the following types of regions:

- one of these regions does not meet the top slice, and so it must be “unoccupied”, and so its only intersection with the bottom slice must be occupied,
- one of these regions meets the top slice in one interval and the bottom in two: thinking of this region as “occupied” gives the generator of type \mathbf{Z} ; thinking of it as unoccupied gives the other two generator types \mathbf{X} and \mathbf{Y} ,
- all other regions meet both the top and the bottom slice in one interval apiece; they can be either occupied or unoccupied.

The bimodules can be graded by $\frac{1}{2}\mathbb{Z}^m$ exactly as in the case of Section 7. The fact that δ_2^1 respects this grading is contained in Lemma 8.1; the fact that δ_1^1 respects it is clear. As in Section 7.1, the grading set of can be thought of as $\frac{1}{2}\mathbb{Z}$ -valued functions on the arcs in the partial knot diagram. This grading is consistent with the grading of \mathcal{K} by $\frac{1}{2}\mathbb{Z}^m$, combined with the induced grading on the tensor product, Proposition 8.4.

9. THE MINIMUM

Fix integers $0 \leq k \leq m+1$ and some $1 \leq c \leq m+1$, and let

$$(9.1) \quad \mathcal{B}_1 = \mathcal{B}(m+2, k+1, \mathcal{S}_1) \quad \text{and} \quad \mathcal{B}_2 = \mathcal{B}(m, k, \mathcal{S}_2)$$

and where $\mathcal{S}_2 \subset \{1, \dots, m\}$ is arbitrary and

$$\mathcal{S}_1 = \phi_c(\mathcal{S}_2) \cup \{c\} \quad \text{or} \quad \mathcal{S}_1 = \phi_c(\mathcal{S}_2) \cup \{c+1\},$$

where ϕ_c is the function from Equation (7.1). We will describe a bimodule ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^c$ that will correspond to introducing a new minimum (or cup) in the diagram that connects the incoming strands c and $c+1$. (We will be primarily interested in the case where $m = 2n$, $k = n = |\mathcal{S}|$.) Note that we follow the convention that \mathcal{B}_1 is the incoming algebra and \mathcal{B}_2 is the outgoing algebra; thus the notation for \mathcal{B}_1 and \mathcal{B}_2 is opposite to the one used in Section 8.

9.1. Description of the bimodule when $c = 1$. We start by describing ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^c$ when $c = 1$ and $2 \in \mathcal{S}_1$ (and so $1 \notin \mathcal{S}_1$).

A *preferred idempotent state* for $\mathcal{B}_1 = \mathcal{B}(m+2, k+1, \mathcal{S}_2)$ is an idempotent state \mathbf{x} with

$$\mathbf{x} \cap \{0, 1, 2\} \in \{\{0\}, \{2\}, \{0, 2\}\}.$$

We define a map ψ from preferred idempotent states of \mathcal{B}_1 to idempotent states of \mathcal{B}_2 , as follows. Given preferred idempotent state \mathbf{x} for \mathcal{B}_1 , order the components $\mathbf{x} = \{x_1, \dots, x_{k+1}\}$ so that $x_1 < \dots < x_{k+1}$. Define

$$\psi(\mathbf{x}) = \begin{cases} \{0, x_3 - 2, \dots, x_{k+1} - 2\} & \text{if } |\mathbf{x} \cap \{0, 1, 2\}| = 2 \\ \{x_2 - 2, \dots, x_{k+1} - 2\} & \text{if } |\mathbf{x} \cap \{0, 1, 2\}| = 1 \end{cases}$$

Generators of the DA bimodule $\mathcal{U}^1 = {}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^1$ correspond to preferred idempotent states; and its bimodule structure over the idempotent algebras is specified by the property that if \mathbf{x} is such a preferred idempotent state, then the corresponding generator $\mathbf{T}_{\mathbf{x}}$ satisfies

$$\mathbf{I}_{\mathbf{y}} \cdot \mathbf{T}_{\mathbf{x}} \cdot \mathbf{I}_{\mathbf{z}} = \begin{cases} \mathbf{T}_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{z} \text{ and } \mathbf{y} = \psi(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

The bimodule structure is expressed in terms of an oriented graph Γ , which we describe presently. The vertices of Γ correspond to words

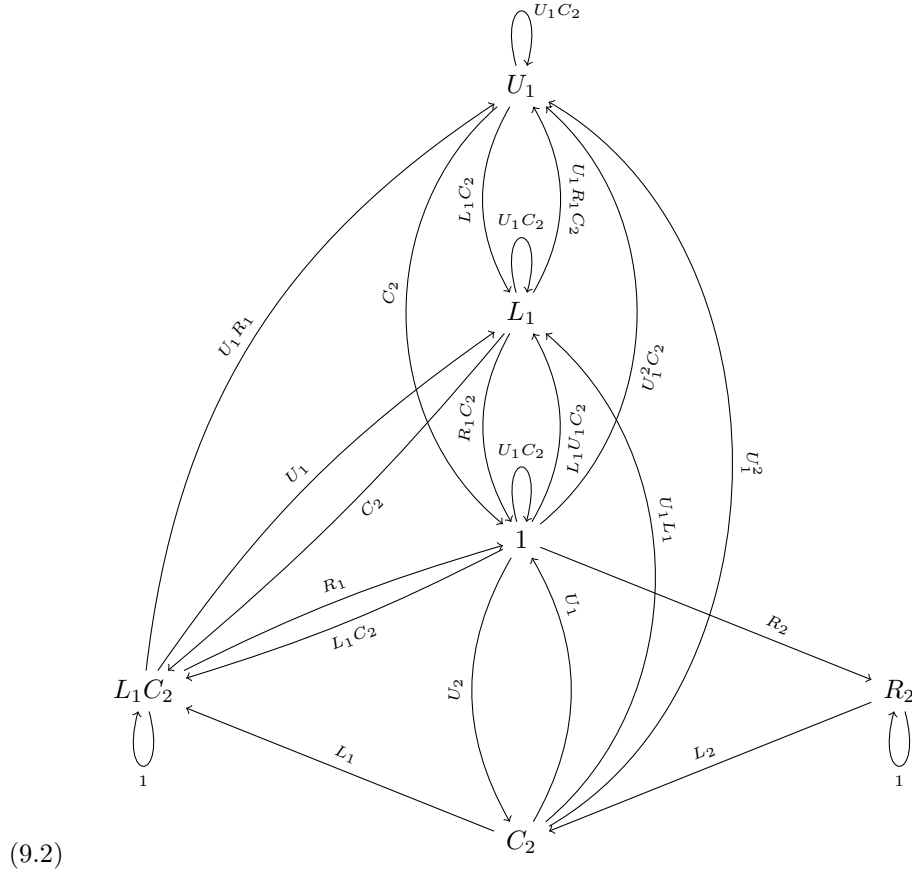
$$\{C_2, L_1 C_2, R_2, U_1^t, L_1 U_1^t\}_{t \geq 0},$$

with the understanding that $U_1^0 = 1$. The graph comes equipped with the following oriented edges, labelled by words of the form $\{1, L_1, U_1^n, U_1^t C_2, R_1 U_1^t C_2\}_{t \geq 0, n > 0}$:

- An edge labelled 1 from R_2 to R_2 .
- An edge labelled 1 from $L_1 C_2$ to $L_1 C_2$.
- An edge labelled $L_1 C_2$ from 1 to $L_1 C_2$.
- An edge labelled L_1 from C_2 to $L_1 C_2$.
- An edge labelled R_2 from 1 to R_2 .
- An edge labelled U_2 from 1 to C_2 .
- An edge labelled L_2 from R_2 to C_2 .
- For each $n > 0$, an edge labelled U_1^n from $L_1 C_2$ to $L_1 U_1^{n-1}$.

- For each $t \geq 0$, an edge labelled $R_1 U_1^t$ from $L_1 C_2$ to U_1^t .
- For each $s \geq 0$ and $t \geq 0$ so that $s + t > 0$, an edge labelled $C_2 U_1^s$ from U_1^t to U_1^{s+t-1} ; and another with the same label from $U_1^t L_1$ to $U_1^{s+t-1} L_1$.
- For each $s \geq 0$ and $t \geq 0$ with $s + t > 0$ an edge labelled $C_2 U_1^s L_1$ from U_1^t to $U_1^{s+t-1} L_1$.
- For each $s \geq 0$ and $t \geq 0$, an edge labelled $C_2 U_1^s R_1$ from $U_1^t L_1$ to U_1^{s+t} .
- For each $n \geq 1$, an edge labelled U_1^n from C_2 to U_1^{n-1} .
- For each $n \geq 1$, an edge labelled $U_1^n L_1$ from C_2 to $U_1^n L_1$.

We illustrate a portion of the graph as follows:



Fix a sequence $a_1, \dots, a_{\ell-1}$ of pure algebra elements in $\mathcal{B}(m+2, k+1, \mathcal{S}_1)$. We call the sequence an *preferred sequence* if the following conditions are satisfied:

- (U-1) there are idempotent states $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ so that $\mathbf{I}_{\mathbf{x}_i} \cdot a_i \cdot \mathbf{I}_{\mathbf{x}_{i+1}} = a_i$ for $i = 1, \dots, \ell - 1$,
- (U-2) The idempotent states \mathbf{x}_1 and \mathbf{x}_ℓ are preferred idempotent states.
- (U-3) Each C_j with $j \in \mathcal{S}_1 \setminus \{2\}$ divides at most one of the a_k .
- (U-4) There is an oriented path e_1, \dots, e_ℓ in Γ beginning at $C_2 L_1$ if $0 \in \mathbf{x}_1$ and at R_2 if $0 \notin \mathbf{x}_1$. The label on the edge e_i , thought of as an algebra element in $\mathcal{B}(m+2, k+1, \mathcal{S}_1)$, has $w_1(a_i) = w_1(e_i)$ and $w_2(a_i) = w_2(e_i)$; and a_i

is divisible by C_2 if and only if e_i is divisible by C_2 , none of the internal vertices are at R_2 or L_1C_2 , and the terminal vertex is at L_1C_2 or R_2 .

For a preferred sequence, there is at most one pure non-zero element $b \in \mathcal{B}_2$ characterized by the following properties

(PS-1) $b = \mathbf{I}_{\psi(\mathbf{x}_1)} \cdot b$

(PS-2) For $j \in \mathcal{S}_2$, C_j divides b if and only if C_{j+2} divides some a_k

(PS-3) For $i = 1, \dots, m$,

$$(9.3) \quad w_i(b) = \sum_{j=1}^{\ell-1} w_{i+2}(a_j).$$

Define maps

$$\delta_\ell^1: \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_1}^1 \otimes \overbrace{\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_1}^{\ell-1} \rightarrow \mathcal{B}_2 \otimes \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_1}^1$$

by specifying them on sequences $a_1, \dots, a_{\ell-1} \in \mathcal{B}(m+2, k+1, \mathcal{S}_1)$ for which there are idempotent states $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ with $\mathbf{I}_{\mathbf{x}_i} \cdot a_i \cdot \mathbf{I}_{\mathbf{x}_{i+1}} = a_i$ and the a_i are pure. For such a sequence, the operation $\delta_\ell^1(\mathbf{T}_{\mathbf{x}_1}, a_1, \dots, a_{\ell-1})$ is non-zero only if the sequence is a preferred sequence, and we define

$$\delta_\ell^1(\mathbf{T}_{\mathbf{x}_1}, a_1, \dots, a_{\ell-1}) = b \otimes \mathbf{T}_{\mathbf{x}_\ell},$$

where b is the algebra element specified by the sequence. For $\ell = 1$, we define $\delta_1^1 = 0$.

For example, sequences $(a_1, \dots, a_{\ell-1})$ for which $\delta_\ell^1(\mathbf{T}, a_1, \dots, a_\ell)$ is non-zero (for suitably chosen \mathbf{T}) include the sequences

$$(9.4) \quad (L_2, L_1), \quad (R_1, R_2), \quad (L_2, U_1, R_2), \quad (U_1, C_2);$$

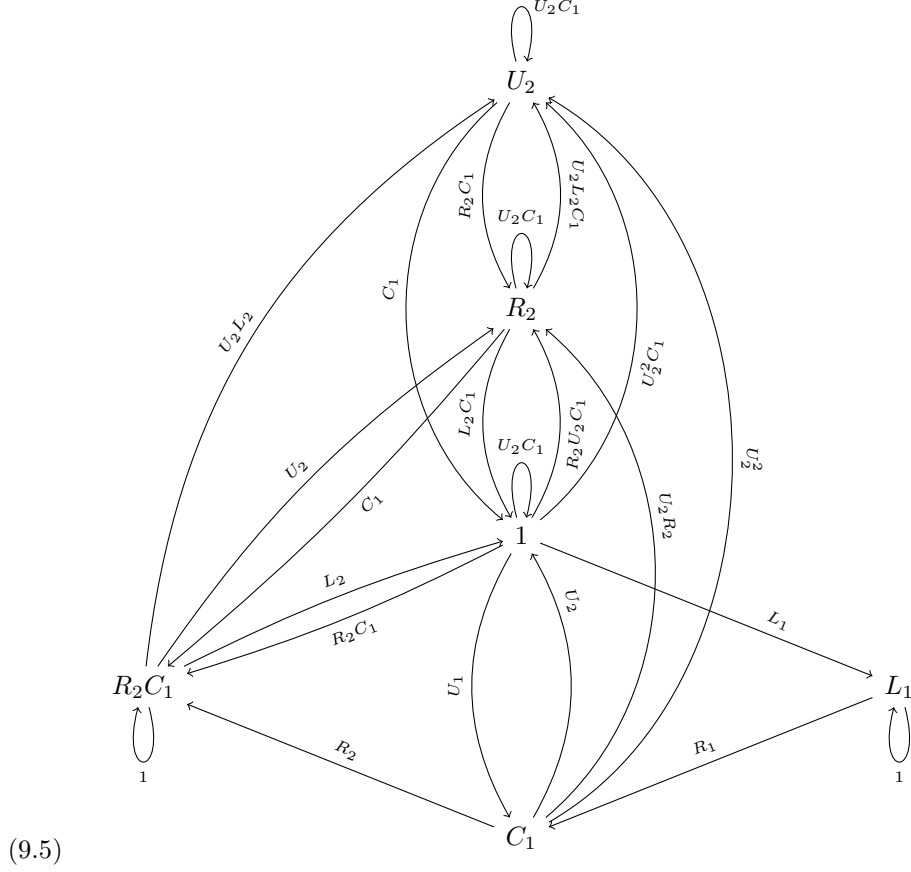
the first comes from a path from R_2 to L_1C_2 ; the second comes from a path from L_1C_2 to R_2 , the third goes from R_2 to itself, and the last goes from L_1C_2 to itself). The loops labelled 1 also give rise to the following actions: if \mathbf{x}_1 and \mathbf{x}_2 are preferred idempotent states, then:

$$\begin{aligned} \delta_2^1(\mathbf{T}_{\mathbf{x}_1}, R_i) &= R_{i-2} \otimes \mathbf{T}_{\mathbf{x}_2} && \text{if } i > 2 \text{ and } \mathbf{I}_{\mathbf{x}_1} \cdot R_i = R_i \cdot \mathbf{I}_{\mathbf{x}_2} \\ \delta_2^1(\mathbf{T}_{\mathbf{x}_1}, L_i) &= L_{i-2} \otimes \mathbf{T}_{\mathbf{x}_2} && \text{if } i > 2 \text{ and if } \mathbf{I}_{\mathbf{x}_1} \cdot L_i = L_i \cdot \mathbf{I}_{\mathbf{x}_2} \\ \delta_2^1(\mathbf{T}_{\mathbf{x}_1}, U_i) &= U_{i-2} \otimes \mathbf{T}_{\mathbf{x}_1} && \text{if } i > 2 \\ \delta_2^1(\mathbf{T}_{\mathbf{x}_1}, C_j) &= C_{j-2} \otimes \mathbf{T}_{\mathbf{x}_1} && \text{if } 2 < j \in \mathcal{S}_1. \end{aligned}$$

As an illustration, we list some other preferred sequences:

$$(U_1 R_1, L_1 C_2, C_2), \quad (L_1, \overbrace{U_1, U_2, \dots, U_1, U_2}^n, L_2), \quad (U_1^n, \overbrace{C_2, \dots, C_2}^n).$$

So far, we described the case where $2 \in \mathcal{S}_1$. In cases where $1 \in \mathcal{S}_1$, we modify the earlier construction slightly as follows: in the description of the graph Γ , switch the roles of U_1 and U_2 , L_1 and R_2 , R_1 and L_2 , C_2 and C_1 . To read off the actions, allowed idempotent states with $0 \in \mathbf{x}$ correspond to the starting vertex L_1 , and those with $0 \notin \mathbf{x}$ correspond to R_2C_1 . With this adjustment, the bimodule is defined as before.



Proposition 9.1. *The above maps δ_ℓ^1 on ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^1$ satisfy the DA bimodule relations; and indeed, if we let W be the collection of arcs in the partial knot diagram with a leftmost minimum, then the bimodule \mathcal{U}^1 is adapted to W , in the sense of Definition 3.18.*

Proposition 9.1 will be proved after we give an alternative construction of \mathcal{U}^1 , later in this section. The significance of \mathcal{U}^1 is formulated in the following:

Lemma 9.2. *\mathcal{U}^1 is dual to \mathcal{E}_1 , in the following sense. Fix arbitrary integers $0 \leq k < m$ and a subsequence $\mathcal{S}_2 = \{u_1, \dots, u_\ell\} \subset \{1, \dots, m\}$, let*

$$\mathcal{S}_1 = \{1, u_1 + 2, \dots, u_\ell + 2\} \quad \text{or} \quad \mathcal{S}_1 = \{2, u_1 + 2, \dots, u_\ell + 2\},$$

$\mathcal{S}_3 = \{1, \dots, m + 2\} \setminus \mathcal{S}_1$, and

$$\mathcal{B}_1 = \mathcal{B}(m + 2, k + 1, \mathcal{S}_1), \quad \mathcal{B}_2 = \mathcal{B}(m, k, \mathcal{S}_2), \quad \mathcal{B}_3 = \mathcal{B}(m + 2, m + 1 - k, \mathcal{S}_3).$$

The following identity holds:

$${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1, \mathcal{B}_3}\mathcal{K} \simeq {}^{\mathcal{B}_2, \mathcal{B}_3}\mathcal{E}_1.$$

Proof. The generators of \mathcal{U}^1 corresponding to idempotent states with $\mathbf{x} \cap \{0, 1, 2\} = \{2\}$, $\{0\}$, and $\{0, 2\}$ respectively induce generators in ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1, \mathcal{B}_3}\mathcal{K}$ which correspond to generators of Ω which, in Section 8, were called of type \mathbf{X} , \mathbf{Y} , and \mathbf{Z} respectively.

If $2 \in \mathcal{S}_1$, we can pair the actions from Equation (9.4) (counting the last one twice) with the differential from \mathcal{K} to give the differentials

$$\begin{aligned} \mathbf{X} &\rightarrow (1 \otimes L_1 L_2) \otimes \mathbf{Y}, & \mathbf{Y} &\rightarrow (1 \otimes R_2 R_1) \otimes \mathbf{X}, & \mathbf{Y} &\rightarrow (1 \otimes C_1 U_2) \otimes \mathbf{Y} \\ \mathbf{X} &\rightarrow (1 \otimes C_1 U_2) \otimes \mathbf{X}, & \mathbf{Z} &\rightarrow (1 \otimes C_1 U_2) \otimes \mathbf{Z} \end{aligned}$$

appearing in the description of ${}^{\mathcal{B}_2, \mathcal{B}_3}\mathcal{E}_1$. The loops labelled 1 induce δ_2^1 actions by the part of the algebra with $w_1 = w_2 = 0$. These actions give rise to the remaining terms for the differential in \mathcal{E}_1 . The case where $1 \in \mathcal{S}_1$ works similarly. \square

9.2. An alternative construction. We describe a $\mathcal{B}_2 - \mathcal{B}_1$ DG bimodule M that is quasi-isomorphic to the bimodule ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^c$ with $c = 1$. Proposition 9.1 will be an immediate consequence of this construction. We continue with the hypothesis that $2 \in \mathcal{S}_1$, returning to the other case at the end of the subsection.

Let $\mathbf{I} = \sum_{\{\mathbf{x} \mid x_1=1\}} \mathbf{I}_{\mathbf{x}} \in \mathcal{B}_1$.

There is a natural inclusion of $\phi: \mathcal{B}_2 \rightarrow \mathbf{I} \cdot \mathcal{B}_1 \cdot \mathbf{I}$, whose image consists of the portion of $\mathbf{I} \cdot \mathcal{B}_1 \cdot \mathbf{I}$ with $w_1 = w_2 = 0$. In particular, $\phi(R_i) = \mathbf{I} \cdot R_{i+2}$ and $\phi(L_i) = \mathbf{I} \cdot L_{i+2}$. Thus, we can think of $\mathbf{I} \cdot \mathcal{B}_1$ as a left module for \mathcal{B}_2 ; it is also a right module for \mathcal{B}_1 . Consider the bimodule

$${}_{\mathcal{B}_2}M_{\mathcal{B}_1} = \mathbf{I} \cdot \mathcal{B}_1 / L_1 L_2 \cdot \mathcal{B}_1,$$

thought of as a $\mathcal{B}_2 - \mathcal{B}_1$ -module, i.e., with $m_{1|1|0}(b, a) = \phi(b) \cdot a$ and $m_{0|1|1}(a, a') = a \cdot a'$; and equipped with the endomorphism $\partial = d + U_1 C_2$.

Lemma 9.3. *The bi-module ${}_{\mathcal{B}_2}M_{\mathcal{B}_1}$ equipped with endomorphism $m_{0|1|0} = \partial$ is isomorphic as left module over \mathcal{B}_2 to a direct sum of*

$${}_{\mathcal{B}_2}M \cong \bigoplus_{a_i} \mathcal{B}_2 \otimes_{\mathbf{I}(\mathcal{B}_2)} a_i,$$

where the a_i are chosen from the generating set

$$(9.6) \quad \{R_2 U_2^t, \quad U_2^n, \quad C_2 R_2 U_2^t, \quad C_2 U_2^t, \quad L_1 U_1^t, \quad U_1^t, \quad C_2 U_1^t\}_{t \geq 0, n > 0}.$$

Proof. Consider any $a = \mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{y}}$ with $x_1 = 1$, and with non-trivial projection to $\mathbf{I} \cdot \mathcal{B}_1 / L_1 L_2 \cdot \mathcal{B}_1$. If $y_1 = 0$, we can write in fact, $a = L_1 U_1^t \cdot a_2$ with $w_1(a_2) = w_2(a_2) = 0$; indeed, in this case, $a = a'_2 L_1 U_1^t$ (where a'_2 and a'_1 differ in their initial idempotent). If $y_1 = 2$, then $a = b \cdot R_2 U_2^t$ or $b \cdot R_2 U_2^t C_2$. Finally, if $y_1 = 1$, then $a = b \cdot U_2^t$ or $a = b \cdot U_1^t$. This shows that the generating sets are as enumerated above.

So far, we have identified the \mathcal{B}_2 -orbits inside of we have shown that

$${}_{\mathcal{B}_2}M \cong \bigoplus_{a_i} \mathcal{B}_2 \cdot a_i,$$

where the a_i are as listed above. For each of the above summands,

$$\mathcal{B}_2 \cdot a_i \cong \mathcal{B}_2 \otimes_{\mathbf{I}(\mathcal{B}_2)} a_i$$

since for \mathbf{x} and \mathbf{y} with $x_1 = 1$ and $\mathbf{x} \cdot \mathcal{B}_2 \cdot \mathbf{y} \neq 0$, the generating intervals that contain 2 also contain 1. \square

According to the above lemma, ${}_{\mathcal{B}_2}M_{\mathcal{B}_1}$ can be thought of as a type DA -bimodule with generating set specified in the lemma (see Equation (2.3)); i.e. there is a type DA bimodule ${}^{\mathcal{B}_2}X_{\mathcal{B}_1}$ so that ${}_{\mathcal{B}_2}M_{\mathcal{B}_1} = \mathcal{B}_2 \boxtimes^{\mathcal{B}_2} X_{\mathcal{B}_1}$. There is a much smaller model for this type DA bimodule using the following:

Lemma 9.4. *The inclusion map $(\mathcal{B}_2 \cdot L_1 C_2) \oplus (\mathcal{B}_2 \cdot R_2) \subset {}_{\mathcal{B}_2}M_{\mathcal{B}_1}$ induces an isomorphism in homology.*

Proof. In view of Lemma 9.3, we can define $H: M \rightarrow M$ by specifying

$$\begin{aligned} H(b_2 \cdot R_2 U_2^n) &= b_2 R_2 U_2^{n-1} C_2 \\ H(b_2 \cdot U_2^n) &= b_2 U_2^{n-1} C_2 \\ H(b_2 \cdot C_2 U_1^n) &= b_2 U_1^{n-1} \\ H(b_2) &= H(b_2 \cdot R_2) = H(b_2 C_2 R_2 U_2^t) = H(b_2 \cdot L_1 U_1^t) \\ &= H(b_2 \cdot C_2 R_2 U_2^t) = H(b_2 \cdot L_1 C_2 U_1^t) = 0 \end{aligned}$$

for all $n > 0$, $t \geq 0$, and $b_2 \in \mathcal{B}_2$ (thought of as the subalgebra of $\mathbf{I} \cdot \mathcal{B}_1 \cdot \mathbf{I}$ with $w_1 = w_2 = 0$). It is straightforward to verify that

$$(9.7) \quad \text{Id} + \partial \circ H + H \circ \partial$$

is projection onto the image of $(\mathcal{B}_2 \cdot L_1 C_2) \oplus (\mathcal{B}_2 \cdot R_2) \subset {}_{\mathcal{B}_2}M_{\mathcal{B}_1}$. For example, if $a = b_2 \cdot C_2 U_1^n$ with $n > 0$, then $\partial \circ H(a) = a + (db_2)U_2^{n-1}$, and $H \circ \partial(a) = (db_2)U_1^{n-1}$; so $(\text{Id} + \partial \circ H + H \circ \partial)(a) = 0$. \square

Proof of Proposition 9.1. We continue with the hypothesis that $2 \in \mathcal{S}_1$. By Lemma 9.4 (and the homological perturbation lemma, in the form of Lemma 2.12) $(\mathcal{B}_2 \cdot C_2 L_1) \oplus (\mathcal{B}_2 \cdot R_2)$ inherits an \mathcal{A}_∞ bimodule structure quasi-isomorphic to M . To this end, note that the maps H appearing in Lemma 9.4 are \mathcal{B}_2 -linear, and so they can be thought of as morphisms of type D structures as in Lemma 2.12.

The induced DA bimodule structure on $(\mathcal{B}_2 \cdot C_2 L_1) \oplus (\mathcal{B}_2 \cdot R_2)$ coincides with the bimodule structure on ${}^{\mathcal{B}_2}\mathcal{U}_{\mathcal{B}_1}^1$. In a little more detail, the labels on the edges $e_1, \dots, e_{\ell-1}$, associated to the algebra elements $a_1, \dots, a_{\ell-1}$ are chosen according to the following rules:

- labels for the possible edges e_1 out of $v_1 = R_1$ or $L_1 C_2$ are chosen so that, according to Lemma 9.3 we have $b_1 \in \mathcal{B}_2$ with $v_1 \cdot a_1 = b_1 \cdot v_2$ (thinking of v_i as certain left \mathcal{B}_2 -module generators of M from Equation (9.6));
- labels for the edges e_i with $1 < i < \ell - 1$ from v_i to v_{i+1} (thought of as certain elements of \mathcal{B}_1) are chosen so that, according to Lemma 9.3, there is $b_i \in \mathcal{B}_2$ with

$$v_i \cdot a_i = b_i \cdot w_i,$$

and $v_{i+1} = H(w_i)$

- For the choices of final edges $e_{\ell-1}$ into (with $v_\ell = R_1$ or $L_1 C_2$), once again the labels were chosen so that for some $b_{\ell-1} \in \mathcal{B}_2$,

$$v_{\ell-1} \cdot a_{\ell-1} = b_{\ell-1} \cdot v_\ell.$$

Thus, as in Lemma 2.12, the induced DA bimodule structure associates to the sequence $a_1, \dots, a_{\ell-1}$, is the product $b = b_1 \cdots b_{\ell-1}$. It is easy to see that this is the output element associated to the pure sequence in our definition of δ_ℓ^1 from Section 9.1

Observe that for all sequences $a_1 \otimes \cdots \otimes a_{\ell-1}$ appearing in the \mathcal{A}_∞ operations,

$$\sum_{i=1}^{\ell-1} w_1(a_i) = \sum_{i=1}^{\ell-1} w_2(a_i).$$

This, together with Equation (9.3) amounts to the statement that \mathcal{U}^i is graded by the set $H^1(W, \partial W)$. Observe also that for each preferred sequence $a_1, \dots, a_{\ell-1}$,

$$\sum_{i=1}^{\ell} \mathfrak{m}_2(a_i) - 2w_2(a_i) = 1 - \ell.$$

(To see this, note that the change in the vertical coordinate of an arrow labelled by an algebra element a in Figure 9.2 is given by $\mathfrak{m}_2(a) - 2w_2(a) - 1$.) This, together with Property (PS-2) ensures that the operations respect a \mathbb{Z} -valued Maslov grading. Obviously, \mathcal{U}^i is finite dimensional. We have thus verified that \mathcal{U}^i is adapted to the one-manifold underlying the partial knot diagram with a left-most minimum.

When $1 \in \mathcal{S}_1$, the above discussion applies with straightforward notational changes, switching the roles of U_1 and U_2 , L_1 and R_2 , R_1 and L_2 , C_2 and C_1 . For example, the analogue of Lemma 9.3 holds, where now a_i are chosen from

$$\{L_1 U_1^t, U_1^n, C_1 L_1 U_1^t, C_1 U_1^t, R_2 U_2^t, U_2^t, C_1 U_2^t\}_{t \geq 0, n > 0}.$$

The analogue of Lemma 9.4 gives a model generated by $(\mathcal{B}_2 \cdot R_2 C_1) \oplus (\mathcal{B}_2 \cdot L_1)$. The inherited \mathcal{A}_∞ structure now is obtained from the graph Γ with the corresponding modification. \square

9.3. The bimodule of a minimum with arbitrary c . Having defined ${}^{\mathcal{B}_2} \mathcal{U}_{\mathcal{B}_1}^c$ for $c = 1$, we can inductively define it for arbitrary c by the relation

$$(9.8) \quad {}^{\mathcal{B}_2} \mathcal{U}_{\mathcal{B}_1}^c = {}^{\mathcal{B}_2} \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes^{\mathcal{B}_4} \mathcal{P}_{\mathcal{B}_3}^c \boxtimes^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_1}^{c-1};$$

with algebras \mathcal{B}_1 and \mathcal{B}_2 chosen as in the beginning of the section; and

$$\mathcal{B}_3 = \mathcal{B}(m+2, k+1, \tau_{c-1}(\mathcal{S}_1)) \quad \text{and} \quad \mathcal{B}_4 = \mathcal{B}(m+2, k+1, \tau_c \circ \tau_{c-1}(\mathcal{S}_1)).$$

See Figure 29 for a picture.

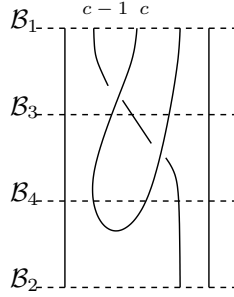


FIGURE 29. To define \mathcal{U}^c , tensor \mathcal{U}^{c-1} with \mathcal{P}^c and \mathcal{P}^{c-1} , as shown.

Proposition 9.5. *For any $1 \leq c \leq m+1$, \mathcal{U}^c is a type DA bimodule that is adapted to the one-manifold underlying its partial knot diagram. Moreover, this bimodule is dual to \mathcal{E}_c , in the sense that*

$$(9.9) \quad \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_1}^c \boxtimes \mathcal{B}_1, \mathcal{B}'_1 \mathcal{K} \simeq \mathcal{B}_2, \mathcal{B}'_1 \mathcal{E}_c,$$

where \mathcal{B}_1 and \mathcal{B}_2 are as in Equation (9.1), and

$$\mathcal{B}'_1 = \mathcal{B}(m+2, m+2-k, \{1, \dots, m+2\} \setminus \mathcal{S}_1).$$

Proof. The tensor product defining \mathcal{U}^c is well defined thanks to Proposition 3.19, and induction on c .

Equation (9.9) is verified by induction on c , and the basic case $c = 1$ is Lemma 9.2. For the inductive step, we compute:

$$\begin{aligned} \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_1}^c \boxtimes \mathcal{B}_1, \mathcal{B}'_1 \mathcal{K} &\simeq \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes \mathcal{B}_4 \mathcal{P}_{\mathcal{B}_3}^c \boxtimes (\mathcal{B}_3 \mathcal{P}_{\mathcal{B}_1}^{c-1} \boxtimes \mathcal{B}_1, \mathcal{B}'_1 \mathcal{K}) \\ &\simeq \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes \mathcal{B}_4 \mathcal{P}_{\mathcal{B}_3}^c \boxtimes (\mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes \mathcal{B}'_3, \mathcal{B}_3 \mathcal{K}) \\ &\simeq \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes (\mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes (\mathcal{B}_4 \mathcal{P}_{\mathcal{B}_3}^c \boxtimes \mathcal{B}_3, \mathcal{B}'_3 \mathcal{K})) \\ &\simeq \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes (\mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes (\mathcal{B}'_3 \mathcal{P}_{\mathcal{B}_4}^c \boxtimes \mathcal{B}'_4, \mathcal{B}_4 \mathcal{K})) \\ &\simeq \mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes \mathcal{B}'_3 \mathcal{P}_{\mathcal{B}_4}^c \boxtimes (\mathcal{B}_2 \mathcal{U}_{\mathcal{B}_4}^{c-1} \boxtimes \mathcal{B}_4, \mathcal{B}'_4 \mathcal{K}) \\ &\simeq \mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes \mathcal{B}'_3 \mathcal{P}_{\mathcal{B}_4}^c \boxtimes \mathcal{B}'_4, \mathcal{B}_2 \mathcal{E}_{c-1} \\ &\simeq \mathcal{B}'_1 \mathcal{P}_{\mathcal{B}_3}^{c-1} \boxtimes \mathcal{B}'_3 \mathcal{N}_{\mathcal{B}_1}^{c-1} \boxtimes \mathcal{B}'_1, \mathcal{B}_2 \mathcal{E}_c \\ &\simeq \mathcal{B}'_1, \mathcal{B}_2 \mathcal{E}_c, \end{aligned}$$

using associativity of \boxtimes (Lemmas 2.4 and 2.9), Lemma 6.6, the trident relation (Lemma 7.2), the inductive hypothesis, and the fact that \mathcal{P} and \mathcal{N} are inverses (Equation (6.1)); these steps are illustrated in Figure 30. (Note that the hypotheses

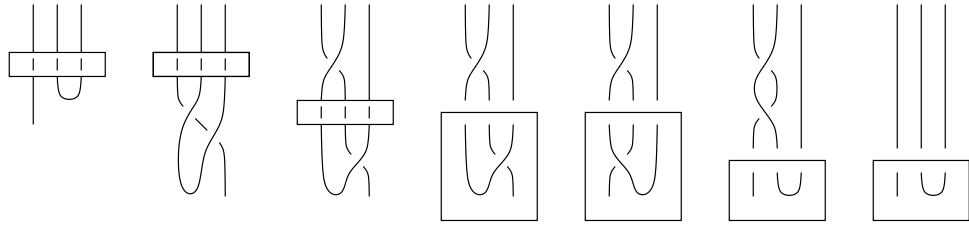


FIGURE 30. **Verifying the inductive step in Proposition 9.1.** Boxed components correspond to type DD bimodules, and unboxed ones correspond to type DA bimodules. Since the order of tensor products is not indicated in these pictures, we have skipped those steps that correspond to associating in different order from the picture.

of Lemma 2.9, are satisfied: the crossing bimodules \mathcal{P}^k for any k have $\delta_j^1 = 0$ for all $j > 3$.) \square

9.4. The terminal Type A module. We have described the DA bimodule associated to a local (but not global) minimum. For the global minimum, we give the following different construction. Note that the algebra immediately above the global minimum has $\mathcal{B} = \mathcal{B}(2, 1, \{2\})$ or $\mathcal{B}(2, 1, \{1\})$, depending on the orientation on the global minimum (left to right or right to left). Consider first the case where $\mathcal{B} = \mathcal{B}(2, 1, \{2\})$. Since the outgoing algebra has no strands (and so it can be thought of as \mathbb{F}), then we construct a type A module over the incoming algebra.

As a vector space, $t\mathcal{U}_{\mathcal{B}(2,1,\{2\})}$ is a two-dimensional, with generators \mathbf{X} and \mathbf{Y} . The right $\mathbf{I}(\mathcal{B}(2, 1, \{2\}))$ -module structure is determined by

$$(9.10) \quad \mathbf{X} = \mathbf{X} \cdot \mathbf{I}_{\{1\}} \quad \mathbf{Y} = \mathbf{Y} \cdot \mathbf{I}_{\{0\}}.$$

The right module structure is determined by the formulas:

$$(9.11) \quad \mathbf{X} \cdot L_1 = \mathbf{Y} \quad \mathbf{Y} \cdot R_1 = \mathbf{X} \quad \mathbf{X} \cdot C_2 = \mathbf{Y} \cdot C_2 = 0.$$

The idempotent relations imply that $\mathbf{X} \cdot R_2 = 0$. It follows that

$$\mathbf{X} \cdot L_1 U_1^t = \mathbf{Y} \quad \mathbf{Y} \cdot R_1 U_1^t = \mathbf{X} \quad \mathbf{X} \cdot U_1^t = \mathbf{X} \quad \mathbf{Y} \cdot U_1^t = \mathbf{Y},$$

for all $t \geq 0$; and U_2 acts trivially.

Defining

$$M(\mathbf{X}) = M(\mathbf{Y}) = 0,$$

it follows that $t\mathcal{U}$ has a \mathbb{Z} -valued Maslov grading.

Unlike the modules encountered thus far, $t\mathcal{U}$ is not graded by Alexander set, which in this case is $\frac{1}{2}\mathbb{Z}$. However, it is filtered by it, in the sense that if a is a homogeneous element with $m_2(X, a) \neq 0$, then $w_2(a) - w_1(a) \leq 0$.

An analogous construction works for $\mathcal{B} = \mathcal{A}(1, \{1\})$. In that case,

$$\mathbf{X} = \mathbf{X} \cdot \mathbf{I}_{\{1\}} \quad \mathbf{Y} = \mathbf{Y} \cdot \mathbf{I}_{\{2\}},$$

and

$$\mathbf{X} \cdot R_2 = \mathbf{Y} \quad \mathbf{Y} \cdot L_2 = \mathbf{X} \quad \mathbf{X} \cdot C_1 = \mathbf{Y} \cdot C_1 = 0.$$

Again, this module is filtered.

Both versions of $t\mathcal{U}$ have an associated graded object, with two generators \mathbf{X} and \mathbf{Y} satisfying Equation (9.10). The actions by $L_1, L_2, R_1, R_2, U_1, U_2$ are all 0; the action by C_1 or C_2 (whichever is in the algebra) is also 0.

Proposition 9.6. *Let Y be a type D structure over $\mathcal{B} = \mathcal{B}(2, 1, \{1\})$ or $\mathcal{B}(2, 1, \{2\})$, with equipped with a \mathbb{Z} -valued Maslov grading and a $\frac{1}{2}\mathbb{Z}$ -valued Alexander grading. Then, the tensor product $t\mathcal{U} \boxtimes Y$ is naturally \mathbb{Z} -graded (by the Maslov grading) and it is filtered by the Alexander grading. Its associated graded object coincides with the tensor product $\widehat{t\mathcal{U}} \boxtimes Y$.*

Proof. The sums in the tensor product are finite since $t\mathcal{U}$ is bounded. The Maslov gradings on Y and $t\mathcal{U}$ induce Maslov gradings on the tensor product as usual, and the Alexander grading on Y induces an Alexander function on $t\mathcal{U} \boxtimes Y$. Since $t\mathcal{U}$ is the associated graded object for $\widehat{t\mathcal{U}}$, it follows that the associated graded object for $t\mathcal{U} \boxtimes Y$ is $\widehat{t\mathcal{U}} \boxtimes Y$. \square

10. SYMMETRIES

We note some symmetries in the algebras and bimodules described above.

One symmetry, which can be visualized as rotation through a vertical axis, is induced by an isomorphism between the algebras. That is, consider the isomorphism $\mathcal{R}: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \rho(\mathcal{S}))$ defined in Equation (3.8). As in Example 2.3, this map gives a DA bimodule, denoted $[\mathcal{R}] = {}^{\mathcal{B}(m, k, \rho(\mathcal{S}))} [\mathcal{R}]_{\mathcal{B}(m, k, \mathcal{S})}$.

The bimodules for crossings and critical points are symmetric under this vertical rotation, according to the following:

Lemma 10.1. *Fix integers c, k , and m , with $1 \leq c \leq m+1$ and $0 \leq k \leq m+1$; and fix $\mathcal{S}_1 \subset \{1, \dots, m\}$, $\mathcal{S}_2 \subset \{1, \dots, m+2\}$ so that $\mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c\}$ or $\mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c+1\}$ (with ϕ_c as in Equation (7.1)). Let $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}_1)$ where $0 \leq k$, and $\mathcal{B}_2 = \mathcal{B}(m, k, \mathcal{S}_2)$, $\mathcal{B}'_1 = \mathcal{B}(m, k, \rho'_m(\mathcal{S}_1))$, $\mathcal{B}'_2 = \mathcal{B}(m, k, \rho'_{m+2}(\mathcal{S}_2))$ (using notation from Section 3.6). The following identities hold:*

$$\begin{aligned} {}^{\mathcal{B}'_2} [\mathcal{R}]_{\mathcal{B}_2} \boxtimes^{\mathcal{B}_2} \Omega_{\mathcal{B}_1}^c &\simeq^{\mathcal{B}'} \Omega_{\mathcal{B}'_1}^{m-c} \boxtimes^{\mathcal{B}'_1} [\mathcal{R}]_{\mathcal{B}_1} \\ {}^{\mathcal{B}'_1} [\mathcal{R}]_{\mathcal{B}_1} \boxtimes^{\mathcal{B}_1} \mathcal{U}_{\mathcal{B}_2}^c &\simeq^{\mathcal{B}'_1} \mathcal{U}_{\mathcal{B}'_2}^{m-c} \boxtimes^{\mathcal{B}'_2} [\mathcal{R}]_{\mathcal{B}'_2} \end{aligned}$$

Also, for any $i = 1, \dots, m-1$, let $\mathcal{B}_3 = \mathcal{B}(m, k, \tau(\mathcal{S}_1))$ and $\mathcal{B}'_3 = \mathcal{B}(m, k, \rho'_m(\tau(\mathcal{S}_1)))$ we have identities

$$\begin{aligned} {}^{\mathcal{B}'_3} [\mathcal{R}]_{\mathcal{B}_3} \boxtimes^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_3}^i &\simeq^{\mathcal{B}'_3} \mathcal{P}_{\mathcal{B}'_1}^{m-i} \boxtimes^{\mathcal{B}'_1} [\mathcal{R}]_{\mathcal{B}_1} \\ {}^{\mathcal{B}'_3} [\mathcal{R}]_{\mathcal{B}_3} \boxtimes^{\mathcal{B}_3} \mathcal{N}_{\mathcal{B}_3}^i &\simeq^{\mathcal{B}'_3} \mathcal{N}_{\mathcal{B}'_1}^{m-i} \boxtimes^{\mathcal{B}'_1} [\mathcal{R}]_{\mathcal{B}_1} \end{aligned}$$

Proof. It is easy to see that $[\mathcal{R}] \boxtimes \mathcal{K} \cong \mathcal{K} \boxtimes [\mathcal{R}]$.

The above results then follow from the invertibility of the DD bimodule, and the easily verified identities

$$[\mathcal{R}] \boxtimes \mathcal{E}_c \cong \mathcal{E}_{m+2-c} \boxtimes [\mathcal{R}]$$

and

$$[\mathcal{R}] \boxtimes \mathcal{P}_i \cong \mathcal{P}_{m-i} \boxtimes [\mathcal{R}],$$

where the latter isomorphism preserves \mathbf{N} and \mathbf{S} and switches \mathbf{E} and \mathbf{W} . \square

Recall that there is an algebra isomorphism

$$o: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \mathcal{S})^{\text{op}},$$

(Equation (3.9)) defined using the identification

$$\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}(m, k, \mathcal{S}) \cdot \mathbf{I}_{\mathbf{y}} \rightarrow \mathbf{I}_{\mathbf{y}} \cdot \mathcal{B}(m, k, \mathcal{S}) \cdot \mathbf{I}_{\mathbf{x}}.$$

An arbitrary DA bimodule ${}^{\mathcal{B}_2} X_{\mathcal{B}_1}$ has an *opposite module* \overline{X} , of the form ${}_{\mathcal{B}_1} X^{\mathcal{B}_2} \cong {}^{\mathcal{B}_2^{\text{op}}} \overline{X}_{\mathcal{B}_1^{\text{op}}}$. If $\mathcal{B}_i = \mathcal{B}(m_i, k_i, \mathcal{S}_i)$ for $i = 1, 2$, we can view the opposite module X^{op} as a bimodule over the same two algebras, $X^{\text{op}} = [o] \boxtimes \overline{X} \boxtimes [o]$.

Proposition 10.2. *Under the identification $\mathcal{M}: \mathcal{B}(m, k, \mathcal{S}) \rightarrow \mathcal{B}(m, k, \mathcal{S})^{\text{op}}$, we have that*

$$\Omega_c^{\text{op}} \simeq \Omega_c \quad \mathcal{U}_c^{\text{op}} \simeq \mathcal{U}_c \quad (\mathcal{P}^i)^{\text{op}} \simeq \mathcal{N}_i \quad (\mathcal{N}^i)^{\text{op}} \simeq \mathcal{P}_i$$

Proof. We claim that $\mathcal{E}_c^{\text{op}} \simeq \mathcal{E}_c$. The identification switches the roles of \mathbf{X} and \mathbf{Y} , and fixes \mathbf{Z} . For each pair of generators in Equation (7.4), there are two arrows, and the symmetry switches those two arrows; observe that if one arrow is labelled by $a \otimes b$, then the other is labelled by $o(a) \otimes o(b)$.

The identity $(\mathcal{P}_i)^{\text{op}} \simeq \mathcal{N}_i$ follows from the definition of \mathcal{N}_i given in Section 5.5. \square

11. CONSTRUCTION AND INVARIANCE OF THE INVARIANT

11.1. Topological preliminaries.

Definition 11.1. A pointed knot diagram is a projection of an oriented knot diagram in S^2 , with a marked point on it.

Definition 11.2. A pointed Reidemeister move is a Reidemeister move disjoint from the supported in a complement of the marked point.

Proposition 11.3. Any two pointed knot diagrams for the same knot can be connected by a sequence of planar isotopies pointed Reidemeister moves.

Proof. According to Reidemeister's theorem [22], any two knot diagrams for the same knot can be connected by a sequence of Reidemeister moves. To see that we can arrange to make those moves disjoint from the marked point p , we argue that p can be moved through a crossing by a sequence of pointed Reidemeister moves, as follows. Suppose we are attempting to move an overcrossing strand across p , situated immediately south of overcrossing. Viewing the knot projection as supported in S^3 , we could equivalently move the overcrossing strand north across the knot diagram, bringing it back north through the point at infinity, as shown in Figure 31. This move can be represented as a sequence of Reidemeister 2 and 3 moves disjoint from p , and it has same effect as moving the overcrossing strand south across p . In this way, we can ensure that all Reidemeister moves are disjoint

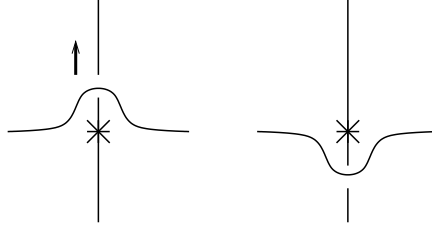


FIGURE 31. Moving an overpass across the basepoint is equivalent to a sequence of Reidemeister moves away from the basepoint.

of p . □

Consider a pointed knot diagram in the plane, and consider the projection to the y axis. We say that the diagram is in *bridge position* if the following properties hold:

- All critical points are all either minima or maxima.
- The minima, maxima, and crossings all project to distinct points in the y axis.
- The global minimum is the marked point.

Definition 11.4. A bridge move connecting two diagrams \mathcal{D}_1 and \mathcal{D}_2 in bridge position is any of the following types of moves:

- Creation of a single local maximum and local minimum; or the cancellation of such a pair. This is called a pair creation or pair annihilation.

- *Sliding a maximum or minimum through a crossing, called a trident move*
- *Commutations between distant pairs of special points (each of which is a maximum, non-global minimum, or a crossing); this is called a commutation move.*

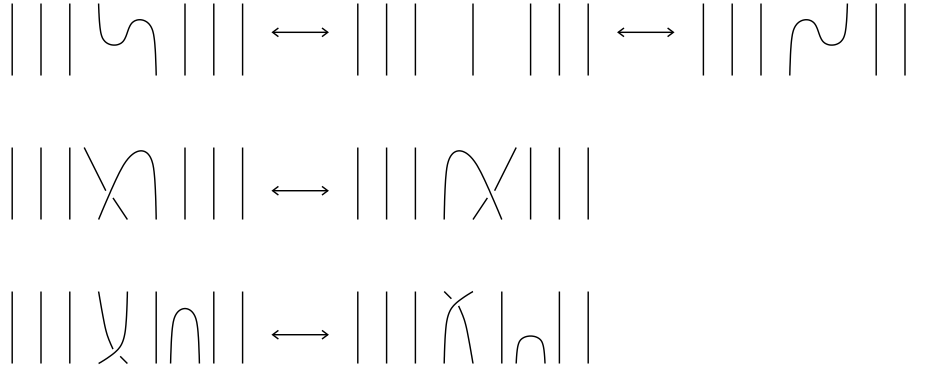


FIGURE 32. **Bridge moves.** The top row illustrates annihilation and creation of a pair of local minimum and a maximum. The second row illustrates a trident move. The three other trident moves are obtained by changing the two crossings, reflecting through a horizontal line, or both. The third row illustrates commutations of a maximum and a distant crossing.

In the above description, “distant” refers to special points not on the same strand, which are covered by the other two kinds of moves.

By adding a point at infinity, a pointed knot diagram in the plane induces a pointed knot diagram in the sphere.

Lemma 11.5. *Any pointed knot diagram is planar isotopic to one in bridge position. Moreover, any two isotopic pointed knot diagrams in bridge position can be connected by a sequence of bridge moves as above.*

Proof. In a codimension one set, the following phenomena can occur: two critical points create or cancel, a crossing contains a critical point, or any other two distant special points project to the same y coordinate. Crossing this codimension one locus, the projection undergoes a pair creation or annihilation, a trident move, or a commutation move.

Thus, a generic pointed knot diagram is in bridge position, and any two such diagrams, when connected by a generic isotopy, are connected by a finite sequence of the above moves. \square

11.2. Constructing the invariant. Choose a knot planar knot diagram \mathcal{D} in bridge position. We slice up the knot diagram into pieces $t_1 < \dots < t_k$ so that the following conditions hold:

- for $i = 1, \dots, k - 1$, the interval $[t_i, t_{i+1}]$ contains the projection onto the y axis of exactly one crossing or critical point

- for $i = 1, \dots, k$, t_i is not the projection of any crossing or critical point
- there are no crossings or critical points whose y value is greater than t_k (and so $[t_{k-1}, t_k]$ contains the global maximum)
- there are no crossings below t_1 , and the only critical point whose y value is smaller than t_1 is the global minimum.

For $i = 1, \dots, k-1$, we have seen how to associate a type DA bimodule to the portion of the diagram that projects into $[t_i, t_{i+1}]$; it is either of the form \mathcal{P}^j , \mathcal{N}^j , Ω^c or \mathcal{U}^c . The top portion $[t_{k-1}, t_k]$ contains the global maximum, where the incoming algebra has no strands, so the DA bimodule is, in fact, simply a type D module (see Section 8.1). The bottom portion $[-\infty, t_1]$ has only the global minimum, so the outgoing algebra has no strands; there we attach the terminal type A module $t\mathcal{U}$ from Section 9.4. Note that for each partial knot diagram, the associated bimodule is adapted to the one-manifold underlying the partial knot diagram; and the hypothesis that our diagram is a indeed one for a knot ensures that the tensor products can always be taken; see Proposition 3.19. It is also worth noting that the generating sets for all the partial knot diagrams correspond to partial Kauffman states; this property is evidently preserved by \boxtimes .

The chain complex $\mathcal{C}(\mathcal{D})$ is now obtained as an iterated tensor product (via \boxtimes) of these pieces. It inherits a filtration, whose associated graded object, $\widehat{\mathcal{C}}(\mathcal{D})$, can be computed by exchanging $t\mathcal{U}$ with the simpler terminal module $\widehat{t\mathcal{U}}$.

Theorem 11.6. *The filtered homotopy type of $\mathcal{C}(\mathcal{D})$ is a knot invariant.*

The quasi-isomorphism type of the chain complex $\mathcal{C}(\mathcal{D})$ does not depend on the order in which the type DA bimodules are tensored together (Lemma 2.4). Thus, to check the invariance of $\mathcal{C}(\mathcal{D})$, it suffices to check identities between DA bimodules associated to the bridge moves, which we do in the next section. The proof of Theorem 11.6 is then given in Subsection 11.4.

11.3. Invariance under bridge moves. We order bridge moves as follows:

- (1) Commutations of distant crossings
- (2) Trident moves
- (3) Critical points commute with distant crossings
- (4) Commuting crossings with critical points
- (5) Commuting distant critical points
- (6) Pair creation and annihilation.

We verify the invariance of our invariant under bridge moves in the above order.

Bridge moves involve the interactions of two consecutive pieces in the chopped up knot diagram. The above procedure associates bimodules to each of those consecutive pieces which we tensor together, and the bridge moves are verified by verifying identities between the bimodule associated to the two bimodules tensored together and the tensor of the two bimodules after the bridge move (except in pair creation or annihilation, which states a relation between the tensor of two bimodules and a third bimodule, the identity bimodule). Thus, for example, commutations of distant positive crossings asserts the relation

$$\mathcal{P}^i \boxtimes \mathcal{P}^j \sim \mathcal{P}^j \boxtimes \mathcal{P}^i$$

when $|i - j| > 1$, which was already verified in the verification of the braid relations (Theorem 6.1).

Consider next trident moves. Note that there are four types of trident moves: the one illustrated in the second row of Figure 32, the one obtained by changing the crossings in the picture, the one obtained by mirroring the picture through a horizontal line, and the one obtained by changing the crossings in the horizontally reflected picture.

Lemma 11.7. *The DA bimodules associated to the two pictures before and after a trident move are quasi-isomorphic; i.e. the following four identities hold, corresponding to the four kinds of trident moves:*

$$(11.1) \quad {}^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_4}^c \boxtimes {}^{\mathcal{B}_4} \Omega_{\mathcal{B}_1}^{c+1} \simeq {}^{\mathcal{B}_3} \mathcal{N}_{\mathcal{B}_2}^{c+1} \boxtimes {}^{\mathcal{B}_2} \Omega_{\mathcal{B}_1}^c$$

$$(11.2) \quad {}^{\mathcal{B}_3} \mathcal{P}_{\mathcal{B}_2}^{c+1} \boxtimes {}^{\mathcal{B}_2} \Omega_{\mathcal{B}_1}^c \simeq {}^{\mathcal{B}_3} \mathcal{N}_{\mathcal{B}_4}^c \boxtimes {}^{\mathcal{B}_4} \Omega_{\mathcal{B}_1}^{c+1}$$

$$(11.3) \quad {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_2}^{c+1} \boxtimes {}^{\mathcal{B}_2} \mathcal{P}_{\mathcal{B}_1}^c \simeq {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_4}^c \boxtimes {}^{\mathcal{B}_4} \mathcal{N}_{\mathcal{B}_1}^{c+1}$$

$$(11.4) \quad {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_2}^c \boxtimes {}^{\mathcal{B}_2} \mathcal{P}_{\mathcal{B}_1}^{c+1} \simeq {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_4}^{c+1} \boxtimes {}^{\mathcal{B}_4} \mathcal{N}_{\mathcal{B}_1}^c$$

Proof. All four identities follow from Lemma 7.2, together with symmetries of the bimodules, as follows.

Since the canonical DD bimodule is invertible (Theorem 3.16), Equation (11.1) follows from the fact that

$$\mathcal{P}^c \boxtimes \Omega^{c+1} \boxtimes \mathcal{K} \simeq \mathcal{N}^{c+1} \boxtimes \Omega^c \boxtimes \mathcal{K};$$

which follows from Proposition 8.4 and Lemma 7.2.

Applying horizontal rotation to Equation (11.1) (with $m - c - 1$ in place of c), together with Lemma 10.1, Equation (11.2) follows from Equation (11.1)

Similarly, observe that

$$\begin{aligned} {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_2}^{c+1} \boxtimes {}^{\mathcal{B}_2} \mathcal{P}_{\mathcal{B}_1}^c \boxtimes {}^{\mathcal{B}_1, \mathcal{B}'_1} \mathcal{K} &\simeq {}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_2}^{c+1} \boxtimes ({}^{\mathcal{B}'_1} \mathcal{P}_{\mathcal{B}'_2}^c \boxtimes {}^{\mathcal{B}'_2, \mathcal{B}_2} \mathcal{K}) \\ &\simeq {}^{\mathcal{B}'_1} \mathcal{P}_{\mathcal{B}'_2}^c \boxtimes ({}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_2}^{c+1} \boxtimes {}^{\mathcal{B}_2, \mathcal{B}'_2} \mathcal{K}) \\ &\simeq {}^{\mathcal{B}'_1} \mathcal{P}_{\mathcal{B}'_2}^c \boxtimes {}^{\mathcal{B}'_2, \mathcal{B}_3} \mathcal{E}_{c+1} \end{aligned}$$

(using Lemma 6.6 twice, Lemma 2.9, Proposition 9.5); and by the same logic,

$${}^{\mathcal{B}_3} \mathcal{U}_{\mathcal{B}_4}^c \boxtimes {}^{\mathcal{B}_4} \mathcal{N}_{\mathcal{B}_1}^{c+1} \boxtimes {}^{\mathcal{B}_1, \mathcal{B}'_1} \mathcal{K} \simeq {}^{\mathcal{B}'_1} \mathcal{N}_{\mathcal{B}'_4}^{c+1} \boxtimes {}^{\mathcal{B}'_4, \mathcal{B}_3} \mathcal{E}_c.$$

Thus, Equation (11.3) follows from Lemma 7.2.

Equation (11.4) follows from Equation (11.3) by horizontal rotation. \square

Lemma 11.8. *The DA bimodules for positive crossings commute with those for local maxima and minima, in the sense that if $|c + 1 - i| > 0$, then*

$$(11.5) \quad \mathcal{P}^i \boxtimes \Omega^c \simeq \Omega^c \boxtimes \mathcal{P}^i$$

$$(11.6) \quad \mathcal{P}^i \boxtimes \mathcal{U}^c \simeq \mathcal{U}^c \boxtimes \mathcal{P}^i$$

Proof. Direct computation, as in the proof of Lemma 6.4, shows that

$$(11.7) \quad \mathcal{P}^i \boxtimes \mathcal{E}_c \simeq \Omega^c \boxtimes \mathcal{P}_i.$$

Both stated equations now follow from Equation (11.7), the invertibility of the canonical type DD bimodule, and Propositions 8.4 and 9.5; compare the proof of Lemma 11.7. \square

Lemma 11.9. *The DA bimodules for arbitrary crossings commute with those for arbitrary distant critical points.*

Proof. Lemma 11.8 handles positive crossings. Multiplying Equation (11.5) on both sides by \mathcal{N}^i and using the fact that \mathcal{N}^i and \mathcal{P}^i are inverses of one another (Equation (6.1)), it follows that negative crossings also commute with local maxima; in the same way, Equation (11.6) shows that negative crossings also commute with local minima. \square

Lemma 11.10. *The DA bimodules associated to pairs of distant critical points commute.*

Proof. Fix $i < j$. The lemma states identifications

$$\begin{aligned}\Omega^i \boxtimes \Omega^{j-1} &\simeq \Omega^{j+1} \boxtimes \Omega^i \\ \mathcal{U}^i \boxtimes \mathcal{U}^{j+1} &\simeq \mathcal{U}^{j-1} \boxtimes \mathcal{U}^i \\ \Omega^{j-1} \boxtimes \mathcal{U}^i &\simeq \mathcal{U}^i \boxtimes \Omega^{j+1} \\ \mathcal{U}^{j+1} \boxtimes \Omega^i &\simeq \Omega^i \boxtimes \mathcal{U}^{j-1}.\end{aligned}$$

To verify the first identity, we tensor on the right with \mathcal{K} , to reduce to the identity between two type DD bimodules (using Proposition 8.4):

$$(11.8) \quad \Omega^i \boxtimes \mathcal{E}_{j-1} \simeq \Omega^{j+1} \boxtimes \mathcal{E}_i,$$

which is easy to verify in the spirit of the proof of Lemma 6.4. Tensoring on the right with \mathcal{K} , the second on the right,

$$\begin{aligned}\mathcal{B}_3 \mathcal{U}_{\mathcal{B}_2}^i \boxtimes \mathcal{B}_2 \mathcal{U}_{\mathcal{B}_1}^{j+1} \boxtimes \mathcal{B}_1 \mathcal{B}'_1 \mathcal{K} &\simeq \mathcal{B}_3 \mathcal{U}_{\mathcal{B}_2}^i \boxtimes \mathcal{B}_2 \mathcal{B}'_1 \mathcal{E}_{j+1} \\ &\simeq \mathcal{B}_3 \mathcal{U}_{\mathcal{B}_2}^i \boxtimes \left(\mathcal{B}'_1 \Omega_{\mathcal{B}'_2}^{j+1} \boxtimes \mathcal{B}_2 \mathcal{B}'_2 \mathcal{K} \right) \\ &\simeq \mathcal{B}'_1 \Omega_{\mathcal{B}'_2}^{j+1} \boxtimes \left(\mathcal{B}_3 \mathcal{U}_{\mathcal{B}'_2}^i \boxtimes \mathcal{B}_2 \mathcal{B}'_2 \mathcal{K} \right) \\ &\simeq \mathcal{B}'_1 \Omega_{\mathcal{B}'_2}^{j+1} \boxtimes \mathcal{B}'_2 \mathcal{B}_3 \mathcal{E}_i\end{aligned}$$

Similarly,

$$\mathcal{U}^{j-1} \boxtimes \mathcal{U}^i \boxtimes \mathcal{K} \simeq \Omega^i \boxtimes \mathcal{E}_{j-1},$$

so the second identity also follows from Equation (11.8).

Consider the third identity, now with $i = 1$. Tensoring on the right with \mathcal{K} , we reduce to the easily verified identity

$$\Omega^{j-1} \boxtimes \mathcal{E}_1 \simeq \mathcal{U}^1 \boxtimes \mathcal{E}_{j+1}.$$

(Compare Equation (11.8).) The cases where $i > 1$ now follow from the case where $i = 1$, the inductive definition of \mathcal{U}^i (Equation (9.8)), and Lemma 11.8.

The fourth identity follows from the third and Lemma 10.1. \square

Lemma 11.11. *The DA bimodules are invariant under pair creation and annihilations, in the sense that $\mathcal{U}^c \boxtimes \Omega^{c+1} \simeq \text{Id} \simeq \mathcal{U}^{c+1} \boxtimes \Omega^c$*

Proof. Rotating through a vertical axis, i.e. using Lemma 10.1, we see that $\mathcal{U}^c \boxtimes \Omega^{c+1} \simeq \text{Id}$ implies also that $\text{Id} \simeq \mathcal{U}^{c+1} \boxtimes \Omega^c$. The verification of $\mathcal{U}^c \boxtimes \Omega^{c+1} \simeq \text{Id}$ can be reduced to the case where $c = 1$ using Reidemeister 2 moves, as follows. First, introduce a sequence of $c - 1$ positive crossings that carry the c^{th} strand to the far left, and let P be the corresponding bimodule; then let N be its inverse, i.e. $N \boxtimes P \simeq \text{Id}$. (Clearly, N is obtained from P by reversing all the crossings and taking them in reverse order.) (cf. Equation (6.1)). In view of Lemma 11.7, a sequence of trident moves identifies $P \boxtimes \mathcal{U}^c \boxtimes \Omega^{c+1} \simeq \mathcal{U}^1 \boxtimes \Omega^2 \boxtimes P$. Thus, if $\mathcal{U}^1 \boxtimes \Omega^2 \simeq \text{Id}$, we can conclude that $\mathcal{U}^c \boxtimes \Omega^{c+1} \simeq N \boxtimes P \boxtimes \mathcal{U}^c \boxtimes \Omega^{c+1} \simeq N \boxtimes \mathcal{U}^1 \boxtimes \Omega^2 \boxtimes P \simeq N \boxtimes P \simeq \text{Id}$.

It remains now to verify that $\mathcal{U}^1 \boxtimes \Omega^2 \simeq \text{Id}$.

We claim that the generators of ${}^{\mathcal{B}_1}\mathcal{U}_{\mathcal{B}_2}^1 \boxtimes {}^{\mathcal{B}_2}\Omega_{\mathcal{B}_1}^2$ correspond to the idempotents in the algebra \mathcal{B}_1 ; see Figure 33. Specifically, $\Omega_{\mathcal{B}}^2$, the generator is constrained to be

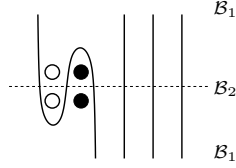


FIGURE 33. **Generators of ${}^{\mathcal{B}_1}\mathcal{U}_{\mathcal{B}_2}^1 \boxtimes {}^{\mathcal{B}_2}\Omega_{\mathcal{B}_1}^2$.** Generators correspond to the idempotents in the incoming algebra \mathcal{B}_1 ; the constraints on the intermediate generators are as shown.

either of type **Y** or **Z**; while in \mathcal{U}^1 , the generator is either of type **X** or **Z**. Thus, in the tensor product, we divide the generators into four types **X** \boxtimes **Y**, **X** \boxtimes **Z**, **Z** \boxtimes **Y**, and **Z** \boxtimes **Z**. We claim that

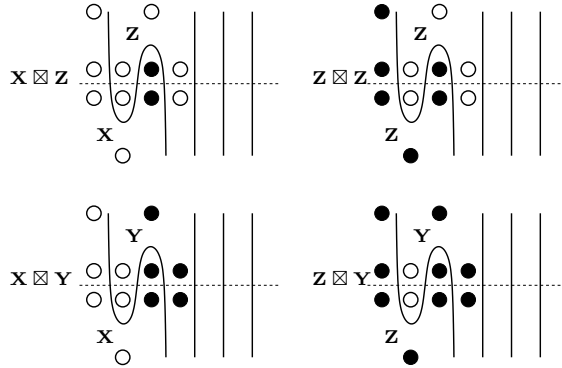
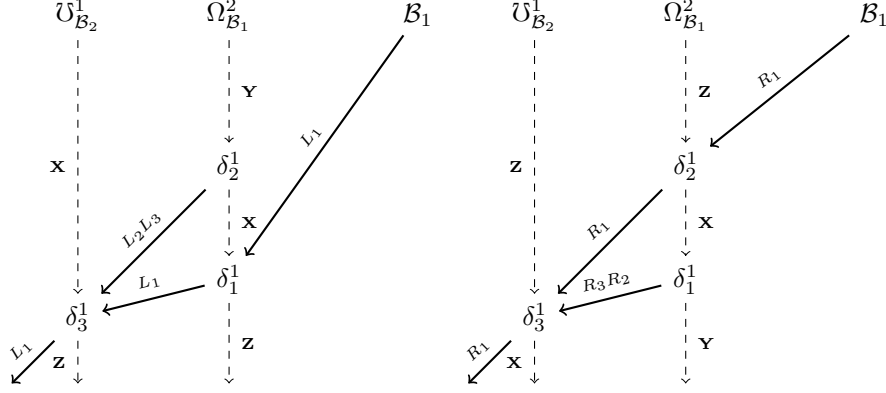


FIGURE 34. **Generators types for ${}^{\mathcal{B}_1}\mathcal{U}_{\mathcal{B}_2}^1 \boxtimes {}^{\mathcal{B}_2}\Omega_{\mathcal{B}_1}^2$.**

$$(11.9) \quad \delta_2^1(\mathbf{X} \boxtimes \mathbf{Y}, L_1) = L_1 \otimes (\mathbf{Z} \boxtimes \mathbf{Z}) \quad \delta_2^1(\mathbf{Z} \boxtimes \mathbf{Z}, R_1) = R_1 \otimes (\mathbf{X} \boxtimes \mathbf{Y}),$$

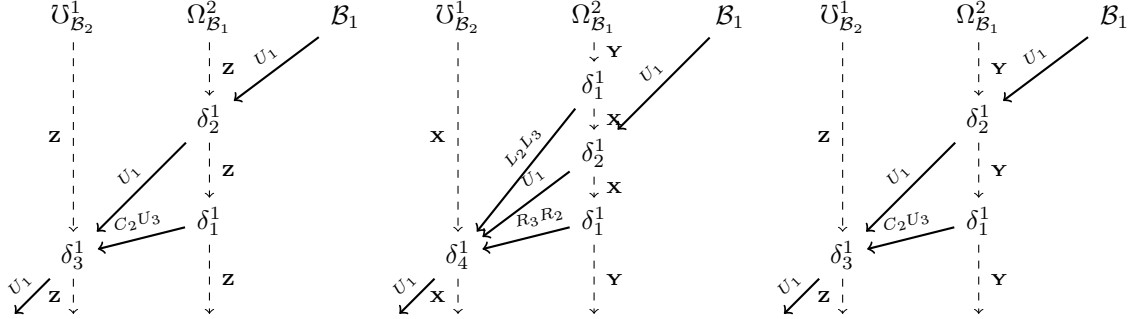
as can be seen from the following diagrams:



If the leftmost strand is downwards, then furthermore

$$(11.10) \quad \delta_2^1(\mathbf{p} \boxtimes \mathbf{q}, U_1) = U_1 \otimes (\mathbf{p} \boxtimes \mathbf{q})$$

for all choices of $\mathbf{p} \in \{\mathbf{X}, \mathbf{Z}\}$ and $\mathbf{q} \in \{\mathbf{Y}, \mathbf{Z}\}$: when $\mathbf{p} \boxtimes \mathbf{q} = \mathbf{X} \boxtimes \mathbf{Z}$, both sides vanish; for the other cases:

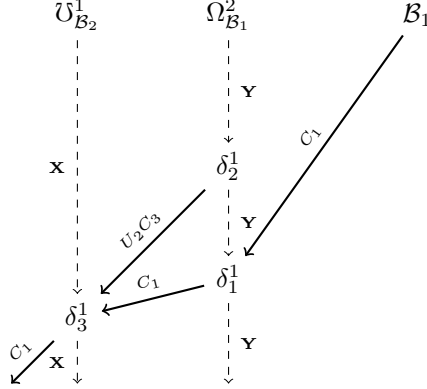


With these computations in hand, it is easy to verify that $U^1 \boxtimes \Omega^2 \boxtimes \mathcal{K} = \mathcal{K}$, and so that $U^1 \boxtimes \Omega^2 \simeq \text{Id}$.

If the leftmost strand is upwards, then instead of Equation (11.10), we claim that

$$\delta_2^1(\mathbf{X} \boxtimes \mathbf{q}, C_1) = C_1 \otimes (\mathbf{X} \boxtimes \mathbf{q})$$

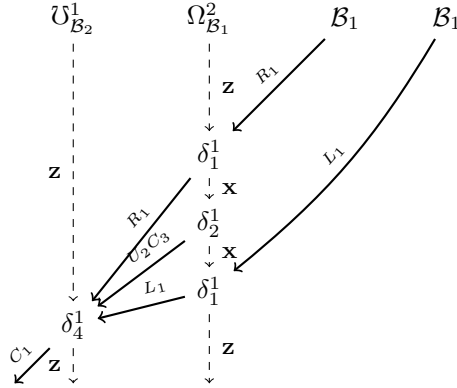
for $\mathbf{q} \in \{\mathbf{Y}, \mathbf{Z}\}$; for example,



We also claim that

$$\delta_3^1(\mathbf{Z} \boxtimes \mathbf{Z}, R_1, L_1) = C_1 \otimes \mathbf{Z} \boxtimes \mathbf{Z},$$

since



Again, it is straightforward to verify now that $\mathcal{U}^1 \boxtimes \Omega^2 \simeq \text{Id}$. \square

11.4. The invariance proof. We can now assemble the pieces to prove Theorem 11.6; and tie it with the discussion from the introduction.

Proof of Theorem 11.6. The tensor product description gives a chain complex thanks to a combination of Proposition 3.19, and finally Proposition 9.6 when attaching the final stage, to get a filtered complex. Combining Proposition 11.3 with the invariance of the bimodules under bridge moves (Theorem 6.1, Lemmas 11.7, 11.8, 11.9, 11.11) shows that (Alexander-)filtered chain homotopy type of $\mathcal{C}(K)$ depends only on the planar diagram for K . (Keeping in mind that homotopy equivalences of DA bimodules induce homotopy equivalences under tensor product.) Invariance under Reidemeister (1) moves follows from the easily checked relation

$$\mathcal{P}_c \boxtimes \Omega_r^c \simeq \Omega_\ell^c.$$

(Here Ω_r^c and Ω_ℓ^c are bimodules associated to maxima, with opposite orientations on the new strand.) Invariance under Reidemeister (2) and (3) moves now follows from the braid relations, Theorem 6.1.

The Alexander filtration, takes values in the H^1 of the knot modulo a neighborhood of the global minimum, can be turned into a rational number by evaluating against the orientation class of the knot. This evaluation is computed by the local formula from Figure 2 according to Proposition 5.16. Since it coincides with the exponent of t in the contribution of the corresponding Kauffman state to the Alexander polynomial (compare [6]), it follows that the Alexander filtration takes values in \mathbb{Z} . \square

Corollary 11.12. *The homology of $\widehat{C}(\mathcal{D})$ is a bigraded knot invariant, whose Euler characteristic agrees with the Alexander polynomial.*

Proof. Invariance follows from Theorem 11.6, since the homology of the associated graded object is invariant under filtered homotopy equivalences. The Euler characteristic computation follows from the fact that generators correspond to Kauffman states; and inspecting Figure 2, we see that $(-1)^{M(\mathbf{x})}t^{A(\mathbf{x})}$ coincides with the monomial associated to a Kauffman state in the computation of the Alexander polynomial from [6]. \square

It is a straightforward matter to go from the filtered homotopy type to the invariant over a polynomial algebra described in the introduction. Explicitly, we replace the filtered terminal type A module $t\mathcal{U}$, with the following variant, which is a module with the following. The same information can be encoded by viewing the terminal minimum $t\mathcal{U}$ as an Alexander graded module $t\mathcal{U}^-$ over a polynomial algebra $\mathbb{F}[v]$; for the case $\mathcal{B}(2, 1\{2\})$ replacing Equation (9.11) with

$$\mathbf{X} \cdot L_1 = v \cdot \mathbf{Y} \quad \mathbf{Y} \cdot R_1 = v \cdot \mathbf{X} \quad \mathbf{X} \cdot C_2 = \mathbf{Y} \cdot C_2 = 0$$

(and making the analogous construction for $\mathcal{B}(2, 1, \{1\})$). In this construction the base algebra now is $\mathbb{F}[v]$, and v is thought of as an element with $A(v) = -1/2$ and $M(v) = -1$. Replacing $t\mathcal{U}$ with $t\mathcal{U}^-$ and forming the same tensor product as before, we arrive at a bigraded complex C^- over $\mathbb{F}[v]$. Since the Alexander grading of generators is integral, we can restrict this to a complex over $\mathbb{F}[U]$ with $U = v^2$. According to Theorem 11.6, the homology of the associated graded object, thought of as a bigraded module over $\mathbb{F}[U]$, is a knot invariant. Setting $v = 0$ (and so $U = 0$) clearly recaptures \widehat{C} . Since U drops Alexander grading by one and Maslov grading by 2, Equation (1.1) follows from the fact that the graded Euler characteristic of \widehat{C} is the Alexander polynomial.

Remark 11.13. *We have described $t\mathcal{U}$ as having two generators. In fact, to define our knot invariants, it suffices to work with the submodule $t\mathcal{U} \cdot \mathbf{I}_{\{1\}}$, since the outputs of our D modules are all contained in the subalgebras of $\mathcal{B}(m, k, \mathcal{S})$*

$$\left(\sum_{\mathbf{x} \mid 0, m \notin \mathbf{x}} \mathbf{I}_{\mathbf{x}} \right) \cdot \mathcal{B}(m, k, \mathcal{S}) \cdot \left(\sum_{\mathbf{x} \mid 0, m \notin \mathbf{x}} \mathbf{I}_{\mathbf{x}} \right).$$

12. FIRST PROPERTIES

We will discuss efficient computations of these invariants in [18]; but there are some easy computations that can be done readily by hand. As a trivial example, the unknot has one-dimensional \widehat{H} , supported in bigrading $(0, 0)$, since it has a diagram with a single Kauffman state in it.

More generally, for an alternating diagram:

Proposition 12.1. *If K is an alternating knot, then $\widehat{H}(K)$ is determined by the signature $\sigma = \sigma(K)$ of K and the symmetrized Alexander polynomial $\Delta_K(t) = \sum_i a_i \cdot t^i$ by*

$$\widehat{H}_d(K, s) = \begin{cases} \mathbb{F}^{|a_s|} & \text{if } d = s + \frac{\sigma}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in [14], this result is simply a consequence of the local contributions to the Alexander and Maslov gradings (pictured in Figure 2). Those formulas show that for an alternating diagram $M - A = \frac{\sigma}{2}$. The rest now follows from the interpretation of the Alexander polynomial in terms of Kauffman states [6]. \square

We list a few of the properties that follow immediately from the constructions from this paper. The methods of this paper are best suited for \widehat{H} ; analogous results for H^- will be established in a follow-up paper [18].

Proposition 12.2. *Let K be a knot and K' denote its mirror. There is an identification $\widehat{H}_d(K, s) \cong \widehat{H}_{-d}(K', -s)$*

Proof. According to Lemma 10.2, the complex of $\widehat{C}(K')$ is computed by tensoring together the opposites of the various bimodules used to compute $\widehat{C}(K)$. It follows that $\widehat{C}(K)$ and $\widehat{C}(K')$ are dual complexes. Since we are working over the field \mathbb{F} , the proposition now follows from the universal coefficient theorem. \square

Proposition 12.3. *Let K_1 and K_2 be two knots. Then, $\widehat{H}(K_1 \# K_2)$ is the bi-graded tensor product of $\widehat{H}(K_1)$ and $\widehat{H}(K_2)$.*

Before turning to the proof, we start with some more general properties.

Let \mathcal{D} be a knot diagram with a single global minimum, oriented up and to the left. Let $\mathcal{B} = \mathcal{B}(2, 1\{1\})$ and let ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ denote the type D structure associated to the diagram with minimum removed.

Lemma 12.4. *Let $\mathcal{B} = \mathcal{B}(2, 1, \{1\})$ and $\mathcal{B}' = \mathbf{I}_{\{1\}} \cdot \mathcal{B} \cdot \mathbf{I}_{\{1\}}$. The output algebra for ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ is contained in \mathcal{B}' ; i.e. if ${}^{\mathcal{B}}[i]_{\mathcal{B}'}$ denotes the bimodule associated to the inclusion $i: \mathcal{B}' \rightarrow \mathcal{B}$, then we have a type D structure ${}^{\mathcal{B}'}\mathcal{C}(\mathcal{D})$ so that*

$${}^{\mathcal{B}}[i]_{\mathcal{B}'} \boxtimes {}^{\mathcal{B}'}\mathcal{C}(\mathcal{D}) = {}^{\mathcal{B}}\mathcal{C}(\mathcal{D}).$$

Proof. For any fixed pair of integers m, k with $0 \leq k \leq m+1$ and $\mathcal{S} \subset \{1, \dots, m\}$, let $\mathcal{B}'(m, k, \mathcal{S}) \subset \mathcal{B}(m, k, \mathcal{S})$, be the subalgebra

$$\mathcal{B}'(m, k, \mathcal{S}) = \left(\sum_{\mathbf{x} \mid 0, m \notin \mathbf{x}} \mathbf{I}_{\mathbf{x}} \right) \cdot \mathcal{B}(m, k) \cdot \left(\sum_{\mathbf{x} \mid 0, m \notin \mathbf{x}} \mathbf{I}_{\mathbf{x}} \right).$$

We claim that all of the DA bimodules Ω^c , \mathcal{U}^c , \mathcal{P}^i , \mathcal{N}^i have the property that their restriction to $\mathcal{B}' \subset \mathcal{B}$ (with appropriate decorations) have their output algebras in the corresponding $\mathcal{B}' \subset \mathcal{B}$. Moreover, the global maximum has output algebra contained in \mathcal{B}' (c.f. Subsection 8.1). \square

Recall that $\mathcal{B}' = \mathbf{I}_{\{1\}} \cdot \mathcal{B}(2, 1, \{1\}) \cdot \mathbf{I}_{\{1\}}$ is $\mathbb{F}[C_1, U_1, U_2]/C_1^2 = 0$, with $dC_1 = U_1$, with gradings

$$(w_1(U_1), w_2(U_1)) = (1, 0) \quad (w_1(C_1), w_2(C_1)) = (1, 0) \quad (w_1(U_2), w_2(U_2)) = (0, 1).$$

It has a subalgebra $\mathcal{B}'' \subset \mathcal{B}'$ isomorphic to $\mathbb{F}[E_1, U_2]/E_1^2 = 0$ (with vanishing differential), and

$$(w_1(E_1), w_2(E_1)) = (1, 1) \quad (w_1(U_2), w_2(U_2)) = (0, 1).$$

The subalgebra is specified by $E_1 = C_1 U_2$.

Lemma 12.5. *With $\mathcal{B} = \mathcal{B}(2, 1, \{1\})$, the type D module ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ is homotopy equivalent to a type D module over $\mathcal{B}'' \subset \mathcal{B}$.*

Proof. Use Lemma 12.4 to restrict to \mathcal{B}' . This result then from homological perturbation theory, and the observation that the inclusion $\mathcal{B}'' \subset \mathcal{B}'$ is a homotopy equivalence. \square

Proof of Proposition 12.3. Consider a connected sum diagram for K_1 and K_2 , where the connected sum region is taken to be the global minimum and the next minimum above it, as pictured in Figure 35.



FIGURE 35. **Connected sums.** Form the disjoint union of the two diagrams on the left; and then .

Taking the disjoint union of K_1 and K_2 (missing the two minima), we obtain a type D structure $\mathcal{C}(K_1 \cup K_2)$ over $\mathcal{B}(4, 2, \{1, 3\})$. Arguing as in Lemma 12.4, the output algebra of this D module is contained in $\mathbf{I}_{\{1, 3\}} \cdot \mathcal{B}(4, 2, \{1, 3\}) \cdot \mathbf{I}_{\{1, 3\}}$. Note that there is a natural identification

$$(\mathbf{I}_{\{1\}} \cdot \mathcal{B}(2, 1, \{1\}) \cdot \mathbf{I}_{\{1\}}) \otimes (\mathbf{I}_{\{1\}} \cdot \mathcal{B}(2, 1, \{1\}) \cdot \mathbf{I}_{\{1\}}) \cong \mathbf{I}_{\{1, 3\}} \cdot \mathcal{B}(4, 2, \{1, 3\}) \cdot \mathbf{I}_{\{1, 3\}};$$

and under this identification,

$$\mathcal{C}(K_1 \cup K_2) = \mathcal{C}(K_1) \otimes \mathcal{C}(K_2).$$

Consider the type A structure N over $\mathcal{B}(4, 2, \{1, 3\})$ with one generator \mathbf{X} satisfying

$$\mathbf{X} \cdot \mathbf{I}_{\{1,3\}} = \mathbf{X},$$

and trivial action by all other pure algebra elements. Clearly,

$$(12.1) \quad N \boxtimes (\mathcal{C}(K_1 \cup K_2)) \cong \widehat{C}(K_1) \otimes \widehat{C}(K_2)$$

Consider next the type A structure P over $\mathcal{B}(4, 2, \{1, 3\})$ obtained by tensoring together a minimum with a global minimum to obtain the type A structure pictured on the right in Figure 35. We restrict to the idempotent type $P \cdot \mathbf{I}_{\{1,3\}}$.

It is easy to see that P has actions $m_2(\mathbf{X}, \mathbf{I}_{\{1,3\}}) = \mathbf{X}$. Moreover, if $m_{\ell+1}(\mathbf{X}, a_1, \dots, a_\ell) \neq 0$, then

$$\sum_{i=1}^{\ell} w_1(a_i) = 0 = \sum_{i=1}^{\ell} w_4(a_i)$$

and

$$(12.2) \quad \sum_{i=1}^{\ell} w_2(a_i) = \sum_{i=1}^{\ell} w_3(a_i).$$

Let

$$\mathcal{B}' = \mathbf{I}_{\{1,3\}} \cdot \mathcal{B}(4, 2, \{1, 3\}) \cdot \mathbf{I}_{\{1,3\}},$$

and \mathcal{B}'' be the subalgebra isomorphic to

$$\mathbb{F}[E_1, E_3, U_2, U_4]/E_1^2 = E_2^2 = 0,$$

were $E_1 = C_1 U_2$ and $E_2 = C_3 U_4$. Lemma 12.5 gives type D structures $\mathcal{B}''^{(2,1,\{1\})} Q_1$ and $\mathcal{B}''^{(2,1,\{1\})} Q_2$ so that

$$\mathcal{B}^{(2,1,\{1\})} \mathcal{C}(K_1) \simeq Q_1 \quad \text{and} \quad \mathcal{B}^{(2,1,\{1\})} \mathcal{C}(K_2) \simeq Q_2;$$

and so

$$\mathcal{B}''(Q_1 \otimes Q_2) \simeq \mathcal{B}' \mathcal{C}(K_1 \cup K_2).$$

Since $w_1(E_1) > 0$, we see that the action by E_1 on P is trivial. Next, we claim that any output algebra element in Q_1 of the form U_2^t with $t \geq 0$ must pair with some action on P with $w_2 > 0$; but such an action must also involve an output from Q_2 with $w_3 > 0$ (by Equation (12.2)), and so it must come from some multiple of E_3 . But any sequence containing a non-trivial factor of E_3 in it acts trivially on P , since $w_4(E_3) > 0$. Similarly, outputs from Q_2 containing a non-trivial multiple of E_3 or U_4 act trivially since w_4 of both of those algebra elements are trivial.

We have shown that the outputs from $Q_1 \otimes Q_2$ giving non-zero differentials when paired with P consist only of the trivial output (1), showing that

$$N \boxtimes (Q_1 \otimes Q_2) \cong P \boxtimes (Q_1 \otimes Q_2) = \widehat{C}(K_1 \# K_2).$$

Combining this with Equation (12.1), we conclude an isomorphism of chain complexes

$$\widehat{C}(K_1 \# K_2) \cong \widehat{C}(K_1) \otimes \widehat{C}(K_2);$$

and the proposition follows readily. \square

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